NOTE ON THE NUMERICAL TRANSCENDENTS S_n AND $s_n = S_n - 1$.

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1. The numbers defined by the series

$$S_n = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots,$$

where n is a positive integer greater than unity, are of frequent occurrence in analysis. Euler, in the Institutiones Calculi Differentialis, 1775, gave a table of their values up to S_{16} to sixteen places of decimals (page 456). He had connected the values of the even-numbered ones with Bernoulli's numbers and the even powers of π by the formula

$$S_{2n} = \frac{2^{2n-1} B_n \pi^{2n}}{(2n)!},$$

but failed to obtain a finite expression for the odd-numbered ones. He gave also the value of the constant γ, now known as Euler's constant, defined as the limiting value

$$\gamma = \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \log n \right]_{n=\infty}.$$

2. The constants γ and S_n occur in the expressions for the function $\log \Gamma(1+x)$; and Legendre, for the purpose of constructing his table of $\log \Gamma(a)$, examined Euler's table of values for S_n and, finding "quelques erreurs assez graves," reconstructed the table, carrying it to S_{35}, remarking that for higher values of n one has only to divide the excess over unity successively by 2. This is of course because the powers of 1/3, 1/4, etc., have disappeared from the last retained decimal place, which was as in Euler's table the sixteenth.

The values of S_n rapidly approach that of the first term, which is unity. It follows that, an algebraic series being given in which the S_n's occur, if the series resulting from replacing S_n by unity has been already summed, we can by subtraction