BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 25, Number 1, July 1991

0273-0979/91 $1.00 + $.25 per page

Propagation and interaction of singularities in nonlinear hyperbolic problems, by Michael Beals. Birkhäuser, Basel, 1989, 143 pp., $29.00. ISBN 3-8176-3449-5

Why should you be interested in the propagation of singularities? The answer is tied up with two other fundamental questions: What is a wave and how do waves serve to send signals? To address all three, begin by considering the classical wave equation $u_{tt} = c^2 u_{xx}$ with $t, x \in \mathbf{R} \times \mathbf{R}$ and $c \in]0, \infty[$. D'Alembert observed that the general solution is $\varphi(x+ct) + \psi(x-ct)$ with φ and ψ arbitrary functions of one variable. The function $\psi(x-ct)$ represents a wave of arbitrary cross-section propagating rigidly at speed c. Similarly $\varphi(x+ct)$ is a wave moving with speed $-c$. Thus the wave equation has two possible modes of propagation, leftward and rightward at speed c.

If one wants to send a message, one could for example represent dot by an upward bump of height 1 and width w and dash by a two humped camel-shaped bump of height 1 and width w and then use the wave equation and Morse code. A similar strategy could be achieved with any equation which transmits two distinct localized waveforms. This is a sort of digital messaging; it is such transmission of information that takes place along axons (the wires in the nervous system) and in computers.

We are miseducated in elementary science courses to believe that sine waves of the form $\sin(k(x \pm ct))$ are what one should think of as waves. Imagine trying to communicate using these. One bump is like any other; there is no beginning and no end. You would like to place a marker on a particular bump and say something like: "A bump with a marker is a dot and one without is a dash." This works, but you can see that the sine wave plays no role; it is just the marker which sends the signal. In summary, for the sake of communication what is useful are localized disturbances which remain recognizable after propagation, and which have understandable laws of motion. Many partial differential equations possess such solutions and may be used in communication.

What distinguishes wave equations, or hyperbolic equations is that they possess an infinite variety of such signals, and in particular signals localized in arbitrarily small regions of space. Thus one