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THE CLASSICAL TRILOGARITHM, ALGEBRAIC K-THEORY OF FIELDS, AND DEDEKIND ZETA FUNCTIONS

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ABSTRACT. In this paper we show how to express the values of $\zeta_F(3)$ for arbitrary number field F in terms of the trilogarithms (D. Zagier's conjecture) and how to relate this result to algebraic K-theory.

1. THE CLASSICAL POLYLOGARITHM FUNCTION

The classical polylogarithm function

$$\mathrm{Li}_p(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^p} (z \in \mathbf{C},\ |z| \leq 1,\ p \in \mathbf{N}) \tag{1.1}$$

during the last 200 years was the subject of much research—see [L]. Using the inductive formula $\mathrm{Li}_p(z) = \int_0^z \mathrm{Li}_{p-1}(t)t^{-1}dt$, $\mathrm{Li}_1(z) = -\log(1-z)$, the p-logarithm can be analytically continued to a multivalued function on $\mathbf{C}\backslash\{0, 1\}$. However, D. Wigner and S. Bloch introduced [B1] the single-valued cousin of the dilogarithm, namely

$$D_2(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\cdot\log|z|. \tag{1.2}$$

Of course, for Li_1 such function is $-\log|z|$. Analogous functions $D_p(z)$ for $p \geq 3$ were introduced in [R] and computed explicitly in [Z]. Let us consider the slightly modified function

$$\mathscr{L}_3(z) := \mathrm{Re}\left[\mathrm{Li}_3(z) - \log|z|\cdot\mathrm{Li}_2(z) + \tfrac{1}{3}\log^2|z|\cdot\mathrm{Li}_1(z)\right]. \tag{1.3}$$

Such modified functions were considered also for all p by D. Zagier, A. A. Beilinson and P. Deligne [Z3, Be1]. $\mathscr{L}_3(z)$ is real-analytic on $CP^1\backslash\{0, 1, \infty\}$ and continuous on CP^1.

Let F be a field. Let P_F^1 be the projective line over F, and let $\mathbf{Z}[P_F^1\backslash 0, 1, \infty]$ be the free abelian group generated by symbols $\{x\}$, where $x \in P_F^1\{0, 1, \infty\}$.

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