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ZERO-ORDER PERTURBATIONS OF THE SUBELLIPTIC LAPLACIAN ON THE HEISENBERG GROUP AND THEIR UNIQUENESS PROPERTIES

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INTRODUCTION

The Heisenberg group $\mathbf{H}^n$ is the step-two nilpotent Lie group whose underlying manifold is $\mathbf{R}^{2n+1}$ equipped with the group law

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(x' \cdot y - x \cdot y')),$$

where $x \cdot y$ denotes the usual inner product in $\mathbf{R}^n$. A basis for the Lie algebra of left-invariant vector fields on $\mathbf{H}^n$ is given by

$$(1) \qquad X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad j = 1, \dots, n, \text{ and } \frac{\partial}{\partial t}.$$

It is readily recognized that $[X_j, Y_k] = -4\delta_{jk}(\partial/\partial t)$ so that, in virtue of a fundamental result of Hörmander [H], the *Kohn-Laplacian*

$$(2) \qquad \Delta_{\mathbf{H}^n} = \sum_{j=1}^{n} (X_j^2 + Y_j^2)$$

is a second-order hypoelliptic (but not elliptic) operator on $\mathbf{H}^n$. In fact, see (7) below, $\Delta_{\mathbf{H}^n}$ is real analytic-hypoelliptic, and therefore a solution to $\Delta_{\mathbf{H}^n} u = 0$ cannot vanish with all its derivatives at one point of a connected open set, unless $u \equiv 0$ in that set.

We are interested in obtaining a quantitative version of the above uniqueness property for solutions to the equation

$$(3) \qquad -\Delta_{\mathbf{H}^n} u + Vu = 0,$$

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