BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 2, October 1990

THE TENSOR PRODUCT PROBLEM FOR REFLEXIVE ALGEBRAS

JON KRAUS

It was observed by Gilfeather, Hopenwasser, and Larson in [1] that Tomita's commutation formula for tensor products of von Neumann algebras can be rewritten in a way that makes sense for tensor products of arbitrary reflexive algebras. The tensor product problem for reflexive algebras is to decide for which pairs of reflexive algebras this tensor product formula is valid.

Recall that a subalgebra $\mathscr{M}$ of the algebra $B(\mathscr{H})$ of all bounded operators on a Hilbert space $\mathscr{H}$ is said to be a *von Neumann algebra* if it is closed in the weak operator topology, contains the identity operator I, and is self-adjoint (i.e., $A \in \mathscr{M}$ implies $A^* \in \mathscr{M}$). The *commutant* $\mathscr{M}'$ of $\mathscr{M}$ is the set of all operators $B \in B(\mathscr{H})$ such that $BA = AB$ for all $A \in \mathscr{M}$. The commutant of a von Neumann algebra is again a von Neumann algebra. Moreover, it follows from von Neumann's double commutant theorem that a self-adjoint subalgebra $\mathscr{M}$ of $B(\mathscr{H})$ is a von Neumann algebra if and only if $\mathscr{M} = \mathscr{M}''$.

Let $\mathscr{M} \subset B(\mathscr{H})$ and $\mathscr{N} \subset B(\mathscr{K})$ be von Neumann algebras, and let $\mathscr{H} \otimes \mathscr{K}$ denote the Hilbert space tensor product of $\mathscr{H}$ and $\mathscr{K}$. If $A \in \mathscr{M}$ and $B \in \mathscr{N}$, there is a unique operator $A \otimes B$ in $B(\mathscr{H} \otimes \mathscr{K})$ such that $(A \otimes B)(x \otimes y) = Ax \otimes By$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$. The von Neumann algebra generated by $\{A \otimes B | A \in \mathscr{M}$ and $B \in \mathscr{N}\}$ is denoted by $\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$. Tomita's commutation theorem asserts that for any pair of von Neumann algebras $\mathscr{M}$ and $\mathscr{N}$ the following commutation formula is valid:

$$\mathscr{M}' \mathbin{\overline{\otimes}} \mathscr{N}' = (\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})'. \tag{1}$$

A number of results concerning tensor products of von Neumann algebras follow from Tomita's theorem. (See, for example, §IV.5 of [13].)

Received by the editors August 23, 1989 and, in revised form, December 8, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47D25, 46L10.

Research partially supported by grants from the National Science Foundation.