BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 1, July 1990

0273-0979/90 $1.00 + $.25 per page

Interpolation of operators, by Colin Bennett and Robert Sharpley. Pure and Applied Mathematics, vol. 129, Academic Press, Orlando, 1988, xiv+469 pp., $69.95. ISBN 0-12-088730-4

Let us start with two L^p spaces, L^{p_0} and L^{p_1}. A function $f \in L^p$, $p_0 < p < p_1$ can be written as a sum of two functions $f = f_0 + f_1$ with $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$. A linear operator defined on both L^{p_0} and L^{p_1} is therefore defined also on all L^p, $p_0 < p < p_1$, provided that the definition of the operator on functions in the two spaces is compatible, i.e., Tf, for $f \in L^{p_0} \cap L^{p_1}$, does not depend on whether we consider f an element of L^{p_0} or of L^{p_1}. One can then reasonably ask what properties of T on the endpoints, L^{p_0} and L^{p_1}, are transferred to the intermediate L^p. This is the simplest example which conveys the idea of interpolation theory.

Interpolation theory has been vastly generalized beyond the concrete setting described above. It is natural to replace L^p spaces by Banach spaces, but important parts of the theory have also been developed in the setting of quasi-Banach spaces (to accommodate L^p spaces with $p < 1$, and, more importantly weak-L^1 and H^p spaces with $p < 1$) and even to quasi-normed groups, and to quasi-normed semi-groups (to accommodate, say, the class of functions with monotone Fourier coefficients). In some cases the single operator T can be replaced by an analytic family of operators, with important consequences in harmonic analysis, and the two-space framework can be replaced by a family of spaces. All these generalizations are motivated by applications to various areas of analysis, principally harmonic analysis, partial differential equations, and approximation theory. Before we discuss some of these generalizations, let us return to the modest setting of L^p spaces described above.

Consider the Fourier transform of periodic functions: $f^\wedge(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-\text{int}}\,dt$. Bessel's inequality gives us

$$\left(\sum_{-\infty}^{\infty} |f^\wedge(n)|^2\right)^{1/2} \le \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\,dt\right)^{1/2},$$