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RAMANUJAN GRAPHS AND HECKE OPERATORS

ARNOLD K. PIZER

0. Introduction

We associate to the Hecke operator T_p, p a prime, acting on a space of theta series an explicit $p+1$ regular Ramanujan graph G having large girth. Such graphs have high "magnification" and thus have many applications in the construction of networks and explicit algorithms (see [LPS1] and Bien's survey article [B]). In general our graphs do not seem to have quite as large a girth as the Ramanujan graphs discovered by Lubotzky, Phillips, and Sarnak ([LPS1, LPS3]) and independently by Margulis ([M]). However, by varying the T_p and the spaces of theta series, we obtain a much larger family of interesting graphs. The trace formula for the action of the Hecke operators T_{p^r} immediately yields information on certain closed walks in G and in particular on the girth of G. If m is not a prime, we obtain "almost Ramanujan" graphs associated to T_m.

The results of this paper can be viewed as an explicit version of a generalization of a construction of Ihara (see [I] and Theorem 4.1 of [LPS2]). From this viewpoint the connection between our results and those of Lubotzky, Phillips, and Sarnak becomes clearer. Recently, Chung ([C]) and Li ([L]) also constructed Ramanujan graphs associated to certain abelian groups.

1. Graphs

Let G be a multigraph (i.e., we allow loops and multiple edges) with n vertices v_i and edges e_j. A *walk* W on G is an alternating sequence of vertices and edges $v_0 e_1 v_1 e_2 v_2 \dots e_r v_r$ where each edge e_j has endpoints v_{j-1} and v_j. We say W is a walk from v_0 to v_r of *length* r. W is *closed* if and only if $v_r = v_0$. A walk is said to be *without backtracking* (a w.b. walk) if a "point can transverse the walk without stopping and backtracking." The only

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