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*Lagrange and Legendre characteristic classes*, by V. A. Vassiliev. Advanced Studies in Contemporary Mathematics, vol. 3, Gordon and Breach, New York, 1988, x + 268 pp., $98.00. ISBN 2-88124-661-3

**1. Caustics and lagrangians.** Originally, caustics were envelopes of families of light rays. If the light propagates in $\mathbf{R}^n$ from a hypersurface (the source) $S_0$ along the half lines $\gamma_q$ normal to $S_0$, we can consider the set of these light rays as a submanifold $L$ of the tangent bundle $T\mathbf{R}^n$,

$$L = \{(\gamma_q(t), p) | t \geq 0, q \in S_0, \|p\| = 1, p \in (T_q S_0)^\perp\}.$$

The envelope of the rays is obtained very simply from $L$. It is just the image of the singular points of the projection from $L$ onto $\mathbf{R}^n$

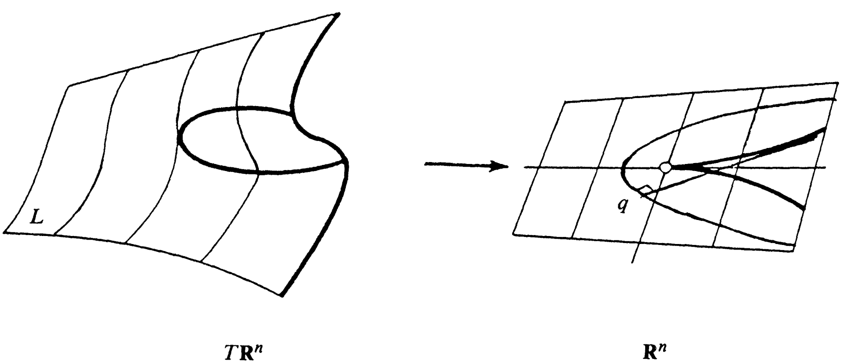


FIGURE 1.

In addition to having the same dimension as $\mathbf{R}^n$, the submanifold $L$ has, by definition, another property: The 1-form $\lambda = p \cdot dq$ is exact when restricted to $L$. (The *optic length* $t$ is a primitive of it.) It is one of the principles of geometric optics that traversing an optical system (lenses, clouds, mirrors, cups of coffee,...) induces a *canonical transformation*: The new manifold of light rays, if it is no longer the manifold of rays orthogonal to a hypersurface, retains these two properties. (See, for example, [**8**].)

In order to generalize these two properties, we replace $\mathbf{R}^n$ by a manifold $X$ and we dispense with the metric, replacing the tangent bundle by the cotangent bundle $T^*X \xrightarrow{\pi} X$, the form $\lambda$ becoming the celebrated *Liouville form* $\lambda = pdq$: If $(q_1, \ldots, q_n)$ are local coordinates on $X$ and $(p_1, \ldots, p_n)$ are the dual coordinates, then $\lambda = \sum_{i=1}^n p_i \, dq_i$.

We consider immersions $f : L \to T^*X$ of manifolds of the same dimension as $X$ such that $f^*\lambda$ is a closed 1-form. We say then that $f$ is *lagrangian*: at each point, the tangent space to $L$ injects as a subspace of the tangent space to $T^*X$ that is maximally totally isotropic for the nondegenerate (symplectic) 2-form $\omega = d\lambda$,

$$d(f^*\lambda) = 0 \iff f^*\omega = 0.$$