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Rational approximation of real functions, by P. P. Petrushev and V. A. Popov. Encyclopedia of Mathematics and its Applications, vol. 28, Cambridge University Press, Cambridge, New York, New Rochelle, Melbourne and Sydney, x + 371 pp., $69.50. ISBN 0-521-33107-2

A book on "approximation theory" can deal with almost any topic. Even "rational approximation" leaves a vast field of mathematics including famous theories like the characterization of those compacts in the complex plane on which good functions can be well approximated by rationals (the Vitushkin theorem, etc.). But *Rational approximation of real functions* makes it quite clear that we are concerned with an important part of the constructive theory of functions in the sense it has been developed mainly by Soviet mathematicians in the tradition of S. N. Bernstein. The book of almost 400 pages by Petrushev and Popov under review gives a stimulating account of the development up to the most recent time of that field, to which both authors have made significant contributions.

The word real in "real functions" is not taken too seriously. It is certainly not easy to completely avoid the complex if you wish to include a chapter on Padé approximation. There is however another reason to involve complex thinking in every discussion of rational approximation (and a very important one according to the reviewer's opinion). Sooner or later you have to comment upon the question if and why rational approximation is "better" than polynomial. The answer is that the rationals are better if the information about the function that shall be approximated is such that you can benefit from your freedom to choose the poles (the pole of a polynomial is quite fixed!). That is true (cf. **[Ga**, Chapter 3]) in the natural generalizations of Zolotarjov's problems from around 1870. (These problems are discussed in the fourth chapter of the book under review.) That is also "the explanation" of the interesting difference between polynomial and rational approximation: How come that in rational approximation the best estimate in the case of a fixed function in some natural class is better than the estimate for the class? The typical example is the following, conjectured by Donald Newman and proved by V. A. Popov.

Let $R_n(f)$ denote the best approximation in the uniform norm by rationals of order n of the function f and let Lip 1 denote the Lipschitz class. Then, $f_1 \in \mathrm{Lip}\,1$ implies that $R_n(f_1) = o(n^{-1})$ but $\sup_{f\in \mathrm{Lip}\,1} R_n(f) \neq o(n^{-1})$.

This is one example of Newman's contributions that revived rational approximation in the sixties. His most famous result is the discovery **[N]** that $|x|$ on $[-1, 1]$ can be uniformly approximated by rationals within $\exp(-c\sqrt{n})$, while the best polynomial approximation is $O(n^{-1})$, not even $o(n^{-1})$.