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Subnormal subgroups of groups, by John C. Lennox and Stewart S. Stonehewer. Oxford Mathematical Monographs, Clarendon Press, Oxford, 1987, xiii + 253 pp., $69.00. ISBN 0-19-853552x

Normality is not a transitive relation for subgroups of a group; the transitive closure of this relation yields subnormality: The subgroup H of the group G is called *subnormal in* G if there exists a finite chain $\{H_i\}$ of subgroups H_i of G with $H = H_0 \leq H_1 \leq \cdots \leq H_i \leq H_{i+1} \leq \cdots \leq H_n = G$ such that H_i is a normal subgroup of H_{i+1} for every $i = 0, \ldots, n-1$. The length of a minimal such chain is the *subnormal defect* of H in G.—In finite groups G the subnormal subgroups are just those subgroups through which passes some composition chain of G; and the Jordan-Hölder theorem states that the set of factors H_{i+1}/H_i of a composition chain of G does not depend on the composition chain chosen. So subnormal subgroups somehow reflect the "normal structure" of the group G. It is almost obvious that the intersection of any two subnormal subgroups of a group G is also subnormal in G. What about the join, i.e. the subgroup of G generated by these two subnormal subgroups? The answer is simple, if the subnormal subgroups H, K are so embedded in G that they permute, i.e., $HK = KH = \langle H, K\rangle$, in this case the subgroup HK is subnormal in G. This observation motivates the search for permutable (subnormal) subgroups of a group. Picking up the above question of Remak's for groups with a (finite) composition chain, in particular for finite groups, H. Wielandt proved in 1939 that the join of two subnormal subgroups is again subnormal. So in finite groups the subnormal subgroups form a sublattice of the lattice of all subgroups. That this is not so for groups in general was first pointed out by Zassenhaus; further examples were produced by P. Hall, Derek Robinson, J. Roseblade and St. Stonehewer....

Wielandt proved in his work that, under his restrictions, a perfect subnormal subgroup H, i.e. one which is generated by the set of its commutators, permutes with every other subnormal subgroup. So the inner structure of H reflects on the way H is embedded in G. J. Roseblade found in 1964 another proof of this result and a sweeping generalisation: If H and K are any two groups, consider their maximal abelian factor groups H/H', K/K' and their tensor product (over $\mathbf{Z}$) $H/H' \otimes K/K'$. These groups are called *orthogonal* (to each other) if $H/H' \otimes K/K' = (0)$. In particular, if H is perfect, then H/H' is trivial and H is orthogonal to every group. Roseblade showed that if two subnormal subgroups H and K of a group G are orthogonal, then they permute and hence their join HK is subnormal in G. Roseblade also gave a converse to this result: If the groups H and K are not orthogonal, then there is a group G with subnormal subgroups isomorphic to H and K which do not permute.

These two results set the themes for much subsequent research: What structural properties of a group G ensure that the join of any two subnormal subgroups of G is again subnormal? Which inner properties of the subnormal subgroups H and K of the group G ensure that $\langle H, K\rangle$ is