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Toposes and local set theories: An introduction, by J. L. Bell. Oxford Logic Guides: 14, Clarendon Press, Oxford, 1988, xii + 267 pp., $75.00. ISBN 0-19-853274-1

It has recently become evident that two apparently different formulations of the foundations of mathematics are merely opposite sides of the same coin. The first of these is the theory of types, going back to Russell and Whitehead in their monumental attempt to rescue Frege from paradox, while the second is the theory of categories, invented by Eilenberg and Mac Lane and conceived as the appropriate language for the foundations of mathematics by Lawvere.

The theory of types, or higher order logic, is called *local set theory* by Bell. As he puts it "types may be thought of as *natural kinds* or *species* from which sets are extracted as subspecies. The resulting theory of sets is *local* in the sense that, for example, the inclusion relation will only obtain among sets which have the same type...."

Unfortunately, the original type theory in Principia Mathematica had proved too cumbersome for most people and, in spite of more elegant formulations by Church and Henkin, was replaced by the set theories of Gödel-Bernays, favoured by mathematicians, and Zermelo-Fraenkel, favoured by logicians. However, in these languages one can ask such meaningless questions as whether the Klein four-group is included in π.

The following simple presentation of type theory had been proposed by Phil Scott and the reviewer **[LS 1983]**. There are given three basic types:

$1 =$ a specified one-entity type introduced for convenience,

$\Omega =$ the type of truth-values or propositions,

$N =$ the type of natural numbers.

From these other types are built up by two operations:

$A \times B =$ the type of pairs of entities of types A and B,

$PA =$ the type of all sets of entities of type A.

In *pure* type theory there will be no other types than those in the hierarchy constructed from the three basic types by the two operations; but in *applied* type theories there may very well be other types, as we shall see later.

A type theory, pure or applied, is a formal language consisting of terms of different types. Among the terms there are countably many variables of each type; we write $x \in A$ to say that x is a variable of type A. From the variables other terms are defined inductively as follows:

$$\begin{array}{ccccc} 1 & \Omega & N & A \times B & PA \\ * & a = a' & 0 & \langle a, b \rangle & \{x \in A | \varphi(x)\} \\ & a \in \alpha & Sn & & \end{array}$$

where it is assumed that a and a' are terms of type A already constructed, α of type PA, n of type N, b of type B and $\varphi(x)$ of type Ω. The usual