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FINITENESS AND VANISHING THEOREMS FOR COMPLETE OPEN RIEMANNIAN MANIFOLDS

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Let M^n denote an n-dimensional complete open Riemannian manifold. In [AG] Abresch and Gromoll introduced a new concept of "diameter growth." Roughly speaking, one would like to measure the essential diameter of ends at distance r from a fixed point $p \in M^n$. They showed that M^n is homotopy equivalent to the interior of a compact manifold with boundary if M^n has nonnegative Ricci curvature and diameter growth of order $o(r^{1/n})$, provided the sectional curvature is bounded from below. It is well known that any complete open manifold with nonnegative sectional curvature has finite topological type. This is a weak version of the *Soul Theorem of Cheeger-Gromoll* [CG]. Examples of Sha and Yang show that this kind of finiteness result does not hold for complete open manifolds with nonnegative Ricci curvature in general (see [**SY1**, **SY2**]), and additional assumptions are therefore required.

We will use a concept of the essential diameter of ends slightly stronger than that of [AG]: For any $r > 0$, let $B(p,r)$ denote the geodesic ball of radius r around p. Let $C(p,r)$ denote the union of all unbounded connected components of $M^n \backslash \overline{B(p,r)}$. For $r_2 > r_1 > 0$, set $C(p;r_1,r_2) = C(p,r_1) \cap B(p,r_2)$. Let $1 > \alpha > \beta > 0$ be fixed numbers. For any connected component Σ of $C(p;\alpha r, \frac{1}{\alpha} r)$, and any two points $x, y \in \Sigma \cap \partial B(p,r)$, consider the distance $d_r(x,y) = \inf \mathrm{Length}(\phi)$ between x and y in $C(p,\beta r)$, where the infimum is taken over all smooth curves $\phi \subset C(p,\beta r)$ from x to y. Set $\mathrm{diam}(\Sigma \cap \partial B(p,r), C(p,\beta r)) = \sup d_r(x,y)$, where $x, y \in \Sigma \cap \partial B(p,r)$. Then the diameter of ends at distance r from p is defined by

$$\mathrm{diam}(p,r) = \sup \mathrm{diam}\left(\Sigma \cap \partial B(p,r), C(p,\beta r)\right),$$

where the supremum is taken over all connected components Σ of $C(p;\alpha r, \frac{1}{\alpha} r)$. The *diameter* defined here is not smaller than that defined by Abresch and Gromoll. Our definition will be essential in Lemma 3 and its applications.

The purpose of this note is to announce the following results.

THEOREM A. *Let M be a complete open Riemannian manifold with sectional curvature $K_M \geq -K^2$ for some constant $K > 0$. Assume that for some base point $p \in M$,*

$$\limsup_{r \to +\infty} \mathrm{diam}(p,r) < \frac{\ln 2}{K}.$$

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