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A NONVANISHING THEOREM FOR DERIVATIVES OF AUTOMORPHIC L-FUNCTIONS WITH APPLICATIONS TO ELLIPTIC CURVES

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1. A brief history of nonvanishing theorems. The nonvanishing of a Dirichlet series $\sum a(n)n^{-s}$, or the existence of a pole, at a particular value of s often has applications to arithmetic. Euler gave the first example of this, showing that the infinitude of the primes follows from the pole of $\zeta(s)$ at $s = 1$. A deep refinement was given by Dirichlet, whose theorem on primes in an arithmetic progression depends in a fundamental way upon the nonvanishing of Dirichlet L-functions at $s = 1$.

Among the many examples of arithmetically significant nonvanishing results in this century, one of the most important is still mostly conjectural. Let E be an elliptic curve defined over $\mathbf{Q}$: the set of all solutions to an equation $y^2 = x^3 - ax - b$ where a, b are rational numbers with $4a^3 - 27b^2 \neq 0$. Mordell showed that $E(\mathbf{Q})$ may be given the structure of a finitely generated abelian group. The *Birch-Swinnerton-Dyer Conjecture* asserts that the rank of this group is equal to the order of vanishing of a certain Dirichlet series $L(s, E)$ as $s = 1$—the center of the critical strip—and that the leading Taylor coefficient of this L-function at $s = 1$ is determined in an explicit way by the arithmetic of the elliptic curve. We refer to the excellent survey article of Goldfeld **[5]** for details.

In 1977, Coates and Wiles **[3]** proved the first result towards the Birch-Swinnerton-Dyer conjecture. The conjecture implies that if the L-series of E does not vanish at 1, then the group of rational points is finite. Coates and Wiles proved this last claim in the special case that E has complex multiplication (nontrivial endomorphisms). In this note, we announce a nonvanishing theorem which, together with work of Kolyvagin and Gross-Zagier, implies that $E(\mathbf{Q})$ is finite when $L(1, E) \neq 0$ for *any* modular elliptic curve E. (A modular elliptic curve is one which may be parametrized by automorphic functions. Deuring proved that all elliptic curves with complex multiplication are modular; Taniyama and Weil have conjectured that indeed all elliptic curves defined over $\mathbf{Q}$ are modular.)

Before giving details of our theorem, we mention several other nonvanishing theorems and arithmetic applications. The following discussion is necessarily not a complete survey

Shimura showed that there is a correspondence between modular forms f of even weight k and modular forms $\tilde{f}$ of half-integral weight $(k+1)/2$.

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