BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 1, July 1989

ITERATION THEORY AND INEQUALITIES FOR KLEINIAN GROUPS

F. W. GEHRING AND G. J. MARTIN

1. Introduction. An important problem in the theory of discrete groups is to decide when two Möbius transformations f, g acting on the Riemann sphere $\overline{\mathbf{C}}$ generate a Kleinian group, that is, a discrete group whose limit set contains more than two points. (See [**Be** and **M1**] for further information on such groups.) Solutions to the above problem have quite general applications, for example, to deformation theory, discreteness of limits [**J1**] and universal constraints for Kleinian groups [**Be**], and lower bounds for the volume of hyperbolic manifolds [**Me**, **W**].

We report here on a connection between this problem and iteration theory [**GM1**]. In particular, by analyzing the stable region D for a certain quadratic polynomial R, we find inequalities which must be satisfied by the generators of a Kleinian group. These include a stronger form of Jørgensen's inequality and inequalities new even for the Fuchsian case.

Our method is similar to that of Zassenhaus [**Z**], Shimizu [**S**], Leutbecher [**Le**], Jørgensen [**J1**], Brooks and Matelski [**BM**] and others. We examine a sequence of subgroups defined by iterating the commutator of the generators; after normalization the traces of the commutators of successive subgroups are related by a quadratic polynomial R. If the trace of the commutator of the original group lies in the region D, then a detailed analysis yields a convergent sequence of elements contradicting discreteness. The main difference in our approach is one of emphasis. Earlier results were obtained by looking for conditions which guarantee the existence of such a sequence. We study the region D and let its geometry dictate what the hypotheses should be.

2. Main result. For $\beta \in \mathbf{C}$ we set $R_\beta(z) = z^2 - \beta z$ and let R_β^n denote the nth iterate of R_β. The filled in Julia set for R_β is the bounded perfect set

$$D(\beta) = \{z \in \mathbf{C}\colon \{R_\beta^n(z)\} \text{ is a bounded sequence}\},$$

and the set of eventually periodic points which do not orbit onto 0 is

$$P^*(\beta) = \{z \in \mathbf{C}\colon \{R_\beta^n(z)\} \text{ is a finite set not containing } 0\}.$$

(See the expository articles of [**Bl** and **Ly**] for further background on this subject.)

Received by the editors July 29, 1988 and, in revised form, January 3, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30F40, 30D50.
The research of both authors was supported in part by the National Science Foundation, Grants DMS-87-02356 and DMS-86-02550.