BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 1, July 1989

CONJECTURE "EPSILON" FOR WEIGHT $k > 2$

BRUCE W. JORDAN AND RON LIVNÉ

Introduction. Let $N, k \geq 1$ be integers and suppose $\varepsilon\colon (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ is a Dirichlet character. Set

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \middle| c \equiv 0 \bmod N \right\}.$$

A holomorphic function f on the Poincaré upper half plane is called a cusp form of level N, weight k, and Nebentypus ε (or briefly a cusp form of type (N, k, ε)) if f has a zero at each cusp and

$$f\left(\frac{az+b}{cz+d}\right) = \varepsilon(d)(cz+d)^k f(z) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

The space of all such cusp forms is denoted $S_k(\Gamma_0(N), \varepsilon)$; this space is acted upon by the Hecke operators $\{T_n, n \geq 1\}$. A function $f \in S_k(\Gamma_0(N), \varepsilon)$ has a Fourier expansion

$$f = \sum_{n\geq 1} a_n(f) q^n = \sum_{n \geq 1} a_n q^n, \qquad q = e^{2\pi i z}.$$

It is said to be normalized if $a_1(f) = 1$. A normalized $f \in S_k(\Gamma_0(N), \varepsilon)$ which is an eigenfunction of all the Hecke operators has $a_n = a_n(f)$ lying in the ring of integers $\mathscr{O} = \mathscr{O}_f$ of the algebraic number field $K_f = \mathbf{Q}(a_n)_{n\geq 1}$.

Consider now a continuous irreducible representation

$$\rho\colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\overline{\mathbf{F}}_l)$$

which is odd, i.e. $\det(\rho(c)) = -1$ where c is complex conjugation. Let $f \in S_k(\Gamma_0(N), \varepsilon)$ be a normalized eigenfunction of the Hecke operators and λ be a prime above l in the extension of K_f generated by the values of ε. Suppose that for each prime p unramified in ρ

$$\mathrm{Tr}(\rho(\mathrm{Frob}_p)) = a_p(f) \mod \lambda$$
$$\det(\rho(\mathrm{Frob}_p)) = \varepsilon(p)p^{k-1} \mod \lambda$$

where $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is a Frobenius element at p. Then we say that ρ arises from f. In **[Se 2]**, Serre conjectures that every such ρ arises from some cusp form f. He furthermore gives a procedure for determining from ρ the type (N, k, ε) of a modular form which gives rise to it. A second conjecture found in **[Se 1]**, implied by the conjecture above, asserts that in certain cases if such a ρ arises from a modular form then it arises from one of the predicted level. This second conjecture was isolated in the case of weight two and dubbed "Epsilon" because it was sufficient due

Received by the editors February 1, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11F33, 14G25.