BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 2, April 1989

0273-0979/89 $1.00 + $.25 per page

Groups and geometric analysis. Integral geometry, invariant differential operators and spherical functions, by Sigurdur Helgason. Academic Press, Orlando, 1984, xviii + 654 pp., $39.50. ISBN 0-12-338301-3

This is a book on harmonic analysis, but harmonic analysis is like the proverbial elephant: it looks very different to different people. To some it means maximal theorems and BMO; to others it means parametrizing the unitary dual; for Harish-Chandra it was the Plancherel Theorem for semisimple groups. Thus no book on harmonic analysis will be universal: each can present only a small part of the subject. To define the book at hand it will help to consider it briefly in its relation to harmonic analysis as a whole, and to examine the types of problems it addresses and does not address.

For purposes of this book, harmonic analysis on a homogeneous space X of a Lie group G means answering the following questions (see p. 2). Write $X = G/K$ for an appropriate closed subgroup K of G.

First, find the algebra $\mathbf{D}(G/K)$ of G-invariant differential operators on X. Then

(B) Describe the spaces of joint eigenfunctions for the operators in $\mathbf{D}(G/K)$;

(A) Decompose "arbitrary" functions on $X = G/K$ into (superpositions of) joint eigenfunctions of $\mathbf{D}(G/K)$; and

(C) Determine on which joint eigenspaces the natural action of G, by translation of functions, is irreducible.

Not all of the book actually fits this formulation, but much of it does. How does a program based on these problems relate to harmonic analysis as a whole?

We first observe that A, B, C imply a decidedly "concrete" stance toward harmonic analysis, as opposed to an "abstract" one. Classification of representations is not a question here. We are dealing with function spaces of a very concrete sort rather than disembodied locally compact spaces, or even, say, sections of vector bundles.

Second, even within very concrete harmonic analysis, one need not restrict one's attention to homogeneous spaces. Fascinating results about a Hamiltonian action of a torus on a symplectic manifold have been discovered recently **[GS1, GS2]**, but these results are trivial if the action is transitive. And the action of a classical group on several copies of its defining vector space (e.g., O_n on $(\mathbf{R}^n)^m$) is the context for a surprisingly rich theory **[Ge, Ho]**.

Third, given that we will work on a homogeneous space, we observe, as does the author, that the formulation A, B, C puts some conditions on G and X: the algebra $\mathbf{D}(G/K)$ must be large enough to be interesting but small enough to be abelian. One much-studied situation **[FS, Gd, HN]**, in