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An introduction to the theory of the Riemann zeta-function, by S. J. Patterson. Cambridge Studies in Advanced Mathematics, vol. 14, Cambridge University Press, Cambridge and New York, 1988, xiii + 156 pp., $34.50. ISBN 0-521-33535-3

Introduction to analytic number theory, by A. G. Postnikov, with an appendix by P. D. T. A. Elliott. Translated by G. A. Kandall. Translations of Mathematical Monographs, vol. 68, American Mathematical Society, Providence, R.I., 1988, vi + 320 pp., $114.00. ISBN 0-8218-4521-7

Problems concerning the distribution of prime numbers go back to antiquity. Solutions to them seem to be elusive. At the age of 17, Gauss conjectured that the number of primes $\leq x$, denoted $\pi(x)$, should be asymptotic to

$$\operatorname{li} x = \int_2^x \frac{dt}{\log t}.$$

In a classical ten page paper of 1858, Riemann introduced the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\Re(s) > 1)$$

as a function of a complex variable and showed how the analytic properties of $\zeta(s)$ should prove the asymptotic law conjectured by Gauss. This was the beginning of the persistent theme of L-series in number theory, the interplay of analysis and arithmetic. The fundamental relation connecting the ζ-function with prime numbers is the "Euler product"

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the product is over primes p. This relation is equivalent to the unique factorization of natural numbers. In his seminal paper, Riemann derived an analytic continuation of $\zeta(s)$ as a meromorphic function of s with only one singularity, a simple pole at $s = 1$. By using the modular transformation of the theta function, he derived the functional equation

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1/s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Without giving any details, he wrote down an infinite product expansion for $\zeta(s)$,

$$s(s-1)\Gamma(s/2)\zeta(s) = e^{bs}\prod_\rho \left(1 - \frac{s}{\rho}\right)e^{s/\rho}$$

where ρ runs through the zeroes of $\zeta(s)$ satisfying $0 \leq \Re(\rho) \leq 1$. In the same spirit, he wrote down an asymptotic formula for the number of such zeroes s satisfying $0 \leq \Im(s) \leq T$. The most astounding idea of the paper