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SYMMETRIC DECREASING REARRANGEMENT CAN BE DISCONTINUOUS

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Suppose $f(x^1, x^2) \geq 0$ is a continuously differentiable function supported in the unit disk in the plane. Its symmetric decreasing rearrangement is the rotationally invariant function $f^*(x^1, x^2)$ whose level sets are circles enclosing the same area as the level sets of f. Such rearrangement preserves $\mathbf{L}^p$ norms but decreases convex gradient integrals, e.g. $\|\nabla f^*\|_p \leq \|\nabla f\|_p$ $(1 \leq p < \infty)$. Now suppose that $f_j(x^1, x^2) \geq 0$ $(j = 1, 2, 3, \ldots)$ is a sequence of infinitely differentiable functions also supported in the unit disk which converge uniformly together with first derivatives to f. The symmetrized functions also converge uniformly. The real question is about convergence of the derivatives of the symmetrized functions. We announce that *the derivatives of the symmetrized functions need not converge strongly*, e.g. it can happen that $\|\nabla f_j^* - \nabla f^*\|_p \not\to 0$ for every p. We further characterize exactly those f's for which convergence is assured and for which it can fail.

The rearrangement map $\mathscr{R}: f \to f^*$ in general dimensions also decreases gradient norms. For this reason alone, rearrangement has long been a basic tool in the calculus of variations and in the theory of those PDE's that arise as Euler-Lagrange equations of variational problems; it permits one to concentrate attention on radial, monotone functions and thereby reduces many problems to simple one dimensional ones. Some examples are (i) the lowest eigenfunction of the Laplacian in a ball is symmetric decreasing; (ii) the body with smallest capacity for a given volume is a ball **[PS]**; (iii) the optimal functions for the Sobolev and Hardy-Littlewood-Sobolev inequalities are symmetric decreasing and can be explicitly calculated **[LE]**. Other examples are given in **[KB]**.

Obviously $\mathscr{R}$ is highly nonlocal, nonlinear, and nonintuitive, but the property of decreasing gradient norms would lead one to surmise that $\mathscr{R}$ is a smoothing operator in some sense. Thus when W. Ni and L. Nirenberg asked, some years ago, whether $\mathscr{R}$ is continuous in the $\mathbf{W}^{1,p}$ topology the answer appeared to be that it should be so (it is easy to prove that $\mathscr{R}$ is always a contraction in $\mathbf{L}^p$). Indeed, by an elegant analysis Coron **[CJ]** proved this in $\mathbf{R}^1$. An affirmative answer to this question would have meant that the mountain-pass lemma could be used to establish spherically symmetric solutions of certain PDE's, and Coron's result led to just such an application **[RS]**. Our result is that $\mathscr{R}$ is not continuous in $\mathbf{W}^{1,p}(\mathbf{R}^n)$ for $n \geq 2$ and it is surprising, to us at least. Since almost all applications

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