THE STATE OF THE SECOND PART
OF HILBERT'S FIFTH PROBLEM

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1. Introduction. In his celebrated address to the 1900 International Congress of Mathematicians, about unsolved problems, Hilbert (**1902**) stated the second part of his fifth problem as follows.

> "Moreover, we are thus led to the wide and interesting field of functional equations which have been heretofore investigated usually only under the assumption of the differentiability of the functions involved. In particular the functional equations treated by Abel (Oeuvres, vol. 1, pp. 1, 61, 389) with so much ingenuity... and other equations occurring in the literature of mathematics, do not directly involve anything which necessitates the requirement of the differentiability of the accompanying functions... In all these cases, then, the problem arises: *In how far are the assertions which we can make in the case of differentiable functions true under proper modifications without this assumption?*" (Hilbert's emphasis).

As in some others in that series of Hilbert problems, the question is not uniquely defined. If, however, we concentrate on Abel's results, as Hilbert did, we can make a definite judgement on how far we got in the direction recommended by Hilbert and that is what I plan to do here.

Abel had four publications and three manuscripts containing functional equations.

We try to look at most, if not all functional equations on which Abel worked and got results. In §2 we deal with two problems for which Abel did *not* suppose differentiability but where recent results still succeeded in reducing the regularity suppositions.

§§3 and 4 are the main part of this survey: it reports on results where the Hilbert program has been carried through: the differentiability conditions used by Abel have been successfully replaced, some quite recently, by (much) weaker conditions.

Finally, in the second part of §4, I speak about a system of two functional equations with which Abel dealt by way of differentation, and which pair has since been reduced to just one of the two equations but where differentiability has not been eliminated yet, so that is still missing to the complete solution of Hilbert's problem.