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Maximum and minimum principles: A unified approach, with applications, by M. J. Sewell. Cambridge University Press, Cambridge, New York, Melbourne, 1987, xvi + 468 pp., $79.50 (cloth), $34.50 (paper). ISBN 0-521-33244-3

It seems likely that every system of equations can be said to give the location of the stationary value of some function: all we need to justify this conjecture is a suitable function space into which the system can be embedded, an appropriate gradient operator and the function! On the one hand this could be a trivial exercise of no great significance, whilst, alternatively, it might involve an elaborate construction in functional analysis. Self-evidently there will be no unique trio of space, gradient and function which will achieve this stationary principle. For many applications there are 'natural' variational formulations based on energy considerations which lead to stationarity principles. However, there are many differential equations which model nonlinear, dissipative systems which have no obvious variational context.

The construction of stationary principles can often be achieved without too much technical difficulty, but it is generally not enough. From the standpoint of applications in Physics and Engineering, further demands are usually required of the construction. In many problems the generating functional is an integral and the application will only become practically interesting if the so-called stationary value is of some physical significance. Further, there may be side constraints to be satisfied; for example, these could be initial or boundary conditions, integral or differential constraints, or inequalities. Even with a complete system, a variational principle which yields only stationarity may be of limited practical value although the principle itself may indicate the fundamental processes at work in the particular application. To achieve a minimum principle, and, consequently, an upper bound for the stationary value we require further structure in the generating functional. Local convexity of the functional in the neighbourhood of its stationary value provides a sufficient condition for a minimum principle; it is a bonus if we can design functionals which have global convexity over a convenient linear space.

Obviously from an estimating point of view, a minimum principle is a one-sided bound with limited practical use. Ideally, both a maximum and a minimum principle are required for numerical estimation purposes, and this is where the notion of the saddle functional plays an important role. Usually the functional is defined on a product of two linear spaces in such a manner that it is convex in one space, concave in the other and stationary where the dual gradients vanish. The advantage of the convex/concave saddle is that certain partial gradient solutions provide upper bounds and others supply lower bounds to the stationary value.

The construction of bounds is not the only purpose of this approach to variational principles. The origins of this subject lie in duality and the