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Eigenvalues and s-numbers by Albrecht Pietsch. Akademische Verlagsgesellschaft Geest and Portig K.-G., Leipzig, and Cambridge University Press, Cambridge, New York and Melbourne, 1987 (Cambridge Studies in Advanced Mathematics, vol. 13) 360 pp., \$59.50. ISBN 0-521-32532-3

From [P1]: "At the turn of this century, I. Fredholm created the determinant theory of integral operators. Subsequently, D. Hilbert developed the theory of bilinear forms in infinitely many unknowns. In 1918 F. Riesz published his famous paper on compact operators (vollstetige Transformationen) which was based on these ideas. In particular, he proved that such operators have an at most countable set of eigenvalues which, arranged in a sequence, tend to zero. *Nothing was said about the rate of this convergence* (emphasis the reviewer's)".

"On the other hand, I. Schur had already observed in 1909 that the eigenvalue sequence of an integral operator induced by a continuous kernel is square summable. This fact indicates that something gets lost within the framework of Riesz theory. The following problem therefore arises:

Find conditions on the operator T that guarantee that the eigenvalue sequence $(\lambda_n(T))$ belongs to a certain subset of c_0 , such as l_r with $0 < r < \infty$."

(Here, c_0 denotes the space of null sequences with the sup norm and l_r the space of r -summable sequences with norm $\|(a_n)\|_r = (\sum_{n=1}^{\infty} |a_n|^r)^{1/r}$).

This is the basic idea of the book under discussion.

Paraphrased from [R]: Cauchy proved that if $f(\lambda) = \lambda^n - t_1 \lambda^{n-1} \dots \pm t_n = 0$ is the characteristic equation of an $n \times n$ matrix A then t_i is the sum of all the principal i -rowed minors of A . Thus the first coefficient t_1 is given by

$$t_1 = a_{11} + a_{22} + \dots + a_{nn}$$

and is called the *trace* of A . Of course t_1 is also the sum of the roots of $f(\lambda)$, i.e., the sum of the eigenvalues of A (counting multiplicities). Thus, in the matrix setting, trace is linear (sum of diagonal elements) and is also the sum of the eigenvalues. All of this extends naturally to finite rank operators on paired linear spaces $[E, E']$: Every finite rank operator $T: E \rightarrow E$ can be written

$$Tx = \sum_{i=1}^n f_i(x)x_i, \quad f_i \in E', \quad x_i, x \in E.$$

The number

$$\phi\text{-trace } T = \sum_{i=1}^n f_i(x_i),$$

the "functional trace," is independent of the finite representation.

Letting $F(E)$ denote the finite rank operators on E , ϕ -trace as defined above is a linear functional on $F(E)$. Naturally one seeks topologies on F so that " ϕ -trace" extends to a continuous linear functional on the closure \overline{F} of F . The other natural questions, of course, are