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NUMERICAL ORBITS OF CHAOTIC PROCESSES REPRESENT TRUE ORBITS

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1. Introduction. It is the nature of chaotic processes of the form $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$, where $\mathbf{f}\colon \mathbf{R}^d \to \mathbf{R}^d$, that different orbits starting close together will move apart rapidly. When following an orbit numerically, a common accuracy is about 14 digits. For chaotic systems such as the logistic map (or in two-dimensions, the Hénon map), distances between two nearby orbits on average grow geometrically on every iterate. For example, it is not unusual that the distance roughly doubles on every iterate. At that rate two true orbits starting 10^{-14} apart will be more than 1 unit apart after just 50 iterates: the error will be of the same order of magnitude as the variables themselves. The implication is that roundoff error on just the first step is sufficient to destroy totally the ability to predict just 50 iterates later.

While a numerical orbit will diverge rapidly from the true orbit with the same initial point, there often exists a different true orbit with a slightly different initial point which stays near the noisy orbit for a long time. We have developed rigorous numerical procedures to prove there exists a true orbit which stays near the noisy orbit of a given chaotic process for a long time.

We begin by defining the shadowing property. The term pseudo-orbit is used to describe a numerically generated noisy orbit.

DEFINITION. $\{\mathbf{p}_n\}_{n=a}^{b}$ is a δ_f-pseudo-orbit for $\mathbf{f}$ if $|\mathbf{p}_{n+1} - \mathbf{f}(\mathbf{p}_n)| < \delta_f$ for $a \leq n \leq b$, where $\mathbf{f}$ is a D-dimensional map and $\mathbf{p}$ is a D-dimensional vector representing the dynamical variables. We are particularly interested in the case where a and b are finite.

DEFINITION. A *true orbit* $\{\mathbf{x}_n\}$ satisfies $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$.

DEFINITION. The true orbit $\{\mathbf{x}_n\}$ δ_x-*shadows* $\{\mathbf{p}_n\}$ on $[a, b]$ if $|\mathbf{x}_n - \mathbf{p}_n| < \delta_x$.

For comparison of different dynamical systems, we use the convention that each coordinate has been rescaled to the interval $[-1, 1]$ or the square $[-1, 1] \times [-1, 1]$ before the quantities δ_f and δ_x are evaluated.

2. Two-dimensional shadowing. In this work, we investigate shadowing in two-dimensional diffeomorphisms. The first system to be examined is the Hénon map, which is not uniformly hyperbolic. Anosov and Bowen [**1**, **2**] proved shadowing results for uniformly hyperbolic systems. The systems we study are not hyperbolic. See [**3**, **5**, **6**] for one-dimensional shadowing

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