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THE SIMPLE-ZERO CONJECTURE FOR SUPPORT POINTS IN Σ

Y. J. LEUNG AND G. SCHOBER

Denote by Σ the class of functions $f(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ analytic and univalent in $\Delta = \{z: |z| > 1\}$. Let L be a continuous linear functional defined on the space of functions analytic in Δ, with the topology of locally uniform convergence. Linear combinations of coefficients and, more generally, linear combinations of point evaluations of the function and its derivatives are typical examples of continuous linear functionals. A function f in Σ is called a *support point* of Σ for L if $\operatorname{Re} L(f) \geq \operatorname{Re} L(g)$ for all g in Σ and $\operatorname{Re} L(f) > \operatorname{Re} L(g)$ for some g in Σ. The simple-zero conjecture concerns the geometric structure of the continuum $\Gamma = \mathbf{C} \backslash f(\Delta)$ omitted by a support point f.

The mappings of the class Σ arise, for example, in flow problems past an obstacle. In this sense, the class Σ is more natural than the familiar schlicht class S of normalized univalent functions in $|z| < 1$. While the de Branges theorem asserts that the single function $k(z) = \sum_{n=1}^{\infty} nz^n$ solves the coefficient problem for S, the situation for Σ appears to be more difficult. In particular, a sharp bound for b_n is known only for $1 \leq n \leq 3$. One obstacle seems to be that the extremal functions change with n. Another is their nonelementary nature. In fact, there are relatively few conjectures (cf. [**10**]).

Therefore, it is natural to look first for qualitative information. A number of properties are known, even about support points of Σ. For example, it follows from Schiffer's boundary variation (cf. [**9**]) that the set Γ omitted by a support point f corresponding to a linear functional L consists of analytic arcs lying on trajectories of the quadratic differential $L(1/(f-w))\,dw^2$. Trajectories of $Q(w)\,dw^2$ are arcs $w = w(t)$ on which $Q(w(t))w'(t)^2 > 0$, together with their endpoints. As a function of w, $Q(w) = L(1/(f-w))$ has an analytic extension to some open region containing the omitted set Γ. Thus the structure theory of quadratic differentials (cf. [**11**]) shows that the arcs of Γ are joined to one another only at the zeros of $L(1/(f-w))$. At a zero of order $k \geq 1$, at most $k+2$ arcs may join at a time in a subset of $k+2$ equi-angular directions. For example, the coefficient functional $L(f) = b_3$ leads to a quadratic differential of the form $(w^2 - b_1)\,dw^2$. The omitted set of a support point for this functional forks at each of the two simple zeros at $\pm\sqrt{b_1}$ in three equi-angular directions [**4**]. Further examples of support points can be found in [**1, 5, 6, 10**]. Based on these examples, one might conjecture that the quadratic differential $L(1/(f-w))\,dw^2$ associated with a support point f in Σ can have only simple zeros on the set Γ omitted by f. It is the purpose

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