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*Introduction to complex hyperbolic spaces*, by Serge Lang. Springer-Verlag, New York, Berlin, Heidelberg, 1987, viii + 271 pp., $58.00. ISBN 0-387-96447-9

There is a (possibly apocryphal) story about Émile Picard that I heard as a graduate student. Just returned from attending a conference, Picard excitedly informed his father-in-law, Hermite, of a beautiful problem suggested by some work of Weierstrass that he had heard about there: was it possible for a nonconstant entire function on $\mathbf{C}$ to omit two values? Hermite in turn showed Picard some theorems he had proved while Picard was away involving modular functions, including the $\lambda$-invariant. The story is a useful cautionary tale for young mathematicians—go to a conference and you may pick up a good problem; stay away and you might prove a nice theorem. Picard, this one time, did both: the $\lambda$-invariant gives a holomorphic covering map from the upper half-plane $\mathcal{H}$ or, equivalently, the unit disc $\Delta$, to $\mathbf{C} - \{0,1\}$. Any holomorphic map $f\colon \mathbf{C} \to \mathbf{C} - \{0,1\}$ lifts to a holomorphic map $\tilde{f}\colon \mathbf{C} \to \Delta$. By Liouville's Theorem, such a map must be constant, and therefore so is $f$. The complement of any two distinct points of $\mathbf{C}$ is analytically equivalent to the complement of any other two. Thus Picard showed that no nonconstant entire function on $\mathbf{C}$ can omit two values.

One can think of Picard's Theorem as stating that there are no nonconstant holomorphic maps $\mathbf{C} \to \mathbf{C}P^1 = S^2$ which omit three points. Picard went on to consider the case of nonconstant holomorphic maps $\mathbf{C} \to X = \overline{X} - S$, where $\overline{X}$ is a compact Riemann surface and $S$ is a finite set of points. By the Uniformization Theorem, the universal cover $\widetilde{X}$ of $X$ is either $\mathbf{C}P^1$, $\mathbf{C}$, or $\Delta$. An elementary analysis of what the covering transformations can be revealed

$$\widetilde{X} = \mathbf{C}P^1 \to X = \mathbf{C}P^1,$$
$$\widetilde{X} = \mathbf{C} \to X = \mathbf{C},\ \mathbf{C}^*,\ \text{or } \mathbf{C}/\Lambda \text{ for some lattice } \Lambda.$$

In all other cases, $\widetilde{X} = \Delta$ and the same argument via Liouville's Theorem shows that there are no nonconstant holomorphic maps $f\colon \mathbf{C} \to X$. A convenient shorthand for this result is: For $X = \overline{X} - S$ of dimension one,

$$\text{Picard's Theorem holds for } X \Leftrightarrow \text{The Euler number } \chi(X) < 0.$$

Given $X$ above with $\chi(X) < 0$, there are of course lots of nonconstant holomorphic maps $f\colon \Delta \to X$. Schottky and Landau noticed that the elementary Schwartz Lemma, which says that a holomorphic map $f\colon \Delta \to \Delta$ with $f(0) = 0$ has $|f'(0)| \leq 1$, translates into the following quantitative result: given $p \in X$ and a tangent vector $v \in T_p(X)$, there exists a constant $R$ such that there is no holomorphic map $f$ from the disc $\Delta(r)$ with $r \geq R$ with $f(0) = p$, $f'(0) = v$. It is now natural to define a length function

$$|v|_X = \inf\{1/r \mid \text{There exists a holomorphic map } f\colon \Delta(r) \to X \text{ with } f(0) = p, f'(0) = v\}.$$