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DYSON'S CRANK OF A PARTITION

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1. Introduction. In [**3**], F. J. Dyson defined the rank of a partition as the largest part minus the number of parts. He let $N(m, t, n)$ denote the number of partitions of n of rank congruent to m modulo t, and he conjectured

$$N(m, 5, 5n+4) = \tfrac{1}{5}p(5n+4), \qquad 0 \le m \le 4; \tag{1.1}$$

$$N(m, 7, 7n+5) = \tfrac{1}{7}p(7n+5), \qquad 0 \le m \le 6, \tag{1.2}$$

where $p(n)$ is the total number of partitions of n [**1**, Chapter 1]. These conjectures were subsequently proved by Atkin and Swinnerton-Dyer [**2**].

Dyson [**3**] went on to observe that the rank did not separate the partition of $11n+6$ into 11 equal classes even though Ramanujan's congruence

$$p(11n+6) \equiv 0 \pmod{11} \tag{1.3}$$

holds. He was thus led to conjecture the existence of some other partition statistic (which he called the crank); this unknown statistic should provide a combinatorial interpretation of $\frac{1}{11}p(11n+6)$ in the same way that (1.1) and (1.2) treat the primes 5 and 7.

In [**4**, **5**], one of us was able to find a crank relative to vector partitions as follows:

For a partition π, let $\#(\pi)$ be the number of parts of π and $\sigma(\pi)$ be the sum of the parts of π (or the number π is partitioning) with the convention $\#(\phi) = \sigma(\phi) = 0$ for the empty partition ϕ, of 0. Let

$$V = \{(\pi_1, \pi_2, \pi_3) \mid \pi_1 \text{ is a partition into distinct parts, } \pi_2, \pi_3 \text{ are unrestricted partitions}\}.$$

We shall call the elements of V *vector partitions*. For $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ in V we define the sum of parts, s, a weight, ω, and a crank, r, by

$$s(\vec{\pi}) = \sigma(\pi_1) + \sigma(\pi_2) + \sigma(\pi_3), \tag{1.4}$$

$$\omega(\vec{\pi}) = (-1)^{\#(\pi_1)}, \tag{1.5}$$

$$r(\vec{\pi}) = \#(\pi_2) - \#(\pi_3). \tag{1.6}$$

We say $\vec{\pi}$ is a vector partition of n if $s(\vec{\pi}) = n$. For example, if

$$\vec{\pi} = (5+3+2, 2+2+1, 2+1+1)$$

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