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Foundations of algebraic analysis, by Masaki Kashiwara, Takahiro Kawai, and Tatsuo Kimura. Translated by Goro Kato. Princeton Mathematical Series, vol. 37, Princeton University Press, Princeton, 1986, xii + 254 pp., $38.00. ISBN 0-691-08413-0

"Algebraic analysis" is a term coined by Mikio Sato. It encompasses a variety of algebraic methods to study analytic objects; thus, an "algebraic analyst" would establish some properties of a function or a distribution by investigating some linear partial differential operators which annihilate it. Here is a concrete example: let f be a polynomial in n complex variables $x_1, x_2, \ldots, x_n$; if s is a complex number, with $\mathrm{Re}(s) > 0$, $|f|^s$ is a well-defined continuous function on $\mathbf{C}^n$. Bernstein [**2**] showed that $|f|^s$ extends to a meromorphic function of s with values in distributions on $\mathbf{C}^n$. The key step is the abstract derivation of the following equation:

$$\text{(B-S)} \qquad P\left(x, \overline{x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial \overline{x}}\right) |f|^s = b(s) \cdot |f|^{s-2},$$

where $b(s)$ is a nonzero polynomial in s, and P is a partial differential operator with polynomial coefficients involving both the variables x_i and their complex conjugates $\overline{x}_i$. This gives immediately the desired meromorphic continuation, with poles located at $\lambda - 2, \lambda - 4, \ldots$, for λ a zero of the polynomial b.

(B-S) is the so-called Bernstein-Sato differential equation; $b(s)$, if chosen of the form $s^k + a_{k-1}s^{k-1} + \cdots + a_0$ with k minimal, is the *Bernstein-Sato polynomial.* This polynomial is in general hard to compute [**14, 20**]; its zeroes are related to the singularities of the hypersurface $f = 0$, in particular to the monodromy action on the vanishing cycles [**14**].

The algebraic ideas behind this theorem of Bernstein, as well as subsequent generalizations by Kashiwara [**10**] and Björk [**3**], involve so-called "D-modules", which means: modules over the algebra D of partial differential operators (usually, in the complex domain). For instance, the vector spaces of holomorphic (or meromorphic) functions, or of generalized functions, are D-modules, because differential operators operate on them in the usual way. Inside a huge D-module like the space of generalized functions, one may consider smaller ones. For instance, if h is a generalized function, the space of all $P(h)$, $P \in D$, is a D-module: the D-module generated by h. The basic idea now is that this D-module is of "small size" iff there are many P such that $P(h) = 0$, i.e. there are many linear PDEs satisfied by h. Algebraic analysis includes, in particular, many techniques to show that a given D-module is of "small size".

However, D-modules can never be that small! For instance, the algebra D for the complex plane $\mathbf{C}$ is generated by x and d/dx, with the famous relation $[d/dx, x] = 1$. Now it is easy to see that if a D-module M is nonzero, it has to be *infinite-dimensional.* Indeed, if it were finite-dimensional, by linear algebra, there would be a nonzero polynomial $P(x)$ which annihilates M. Then $[d/dx, P(x)] = P'(x)$ also annihilates M. Iterating the process,