A NOTE ON THE LOCATION OF COMPLEX ZEROS OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

STEVEN B. BANK

1. Introduction. For second-order equations, $w'' + A(z)w = 0$, where $A(z) = a_m z^m + \cdots$ is a polynomial of degree $m \geq 1$, there is a classical result (due jointly to E. Hille, R. Nevanlinna and H. Wittich [10, p. 282]) which determines the possible location of the zeros of any solution $f \neq 0$. The theorem states that for any $\epsilon > 0$, all but finitely many zeros of $f$ lie in the union (for $j = 0, 1, \ldots, m + 1$) of the $\epsilon$-sectors, $|\arg z - \phi_j| < \epsilon$, where $\phi_j = (2\pi j - c)/(m + 2)$ for any choice of $c = \arg a_m$. (The rays $\arg z = \phi_j$ are called "critical rays"). In this paper, we determine the situation for higher-order equations.

(1) $w^{(n)} + a_{n-1}(z)w^{(n-1)} + \cdots + a_1(z)w' + a_0(z)w = 0 \quad (n \geq 2),$

where the $a_j(z)$ are polynomials. As shown in Theorem 1 below ($§3$), an interesting feature of the higher-order case is that the Hille-Nevanlinna-Wittich property (i.e., the existence of finitely many critical rays around which the zeros of any solution $f \neq 0$ must be concentrated) need not hold when $n > 2$. There are equations (e.g. see $§4$ below) which have the property that for any ray, and any $\epsilon$-sector around it, some solution $f \neq 0$ has infinitely many zeros in the $\epsilon$-sector. In Theorem 1, we show that in general either this latter property or the Hille-Nevanlinna-Wittich property holds for a given equation (1), and one can easily determine from the equation which of the two holds. In $§7$, we consider the problem of explicitly determining the critical rays for those equations (1) possessing the Hille-Nevanlinna-Wittich property.

The key tools in the proof of Theorem 1 are asymptotic existence theorems which were proved in [4] and [6] using the Strodt theory [8, 9]. (Details of the proof will appear elsewhere.)

2. Preliminaries. Given an equation (1) where the $a_j(z)$ are any rational functions, we first rewrite the equation (1) in terms of the operator $\theta$ defined by $\theta w = zw'$. (It is easy to prove by induction that for each $m = 1, 2, \ldots,$

(2) $w^{(m)} = z^{-m} \left( \sum_{j=1}^{m} b_{jm} \theta^j w \right),$

Received by the editors June 24, 1987.

1980 Mathematics Subject Classification (1985 Revision). Primary 34A20, 34C10.
Research supported in part by the National Science Foundation (DMS-8420561).