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THE INDECOMPOSABLE K_3 OF FIELDS

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In this note, we describe an extension of Hilbert's Theorem 90 for K_2 of fields to the relative K_2 of semilocal PID's containing a field. Most of the results for K_2 of fields proven in [**M-S** and **S**] then carry over to the relative K_2 of semilocal PID's containing a field, e.g. computation of the torsion subgroup, and the norm residue isomorphism. Applying this to the semilocal ring of $\{0,1\}$ in $\mathbf{A}^1_E$, for a field E, gives a computation of the torsion and co-torsion in

$$K_3(E)^{\mathrm{ind}} := K_3(E)/K_3^M(E).$$

Specifically, we have

(1) The torsion subgroup of $K_3(E)^{\mathrm{ind}}$ is $H^0_{\text{ét}}(E, \mu_\infty^{\otimes 2})$.

(2) $K_3(F, \mathbf{Z}/n)^{\mathrm{ind}} \tilde{\to} H^1_{\text{ét}}(E, \mu_n^{\otimes 2})$ for $(n, \mathrm{char}(E)) = 1$, so

$$\varprojlim_n K_3(E)^{\mathrm{ind}}/l^n \tilde{\to} H^1_{\text{ét}}(E, \mathbf{Z}_l(2)) \quad \text{for } l \text{ prime, } l \neq \mathrm{char}(E).$$

(3) $K_3(E)^{\mathrm{ind}}$ satisfies Galois descent for extensions of degree prime to $\mathrm{char}(E)$.

(4) Bloch's group $B(E)$ is uniquely n-divisible if E contains an algebraically closed field, and $(n, \mathrm{char}(E)) = 1$.

The results (3) and (4) follow directly from (1) and (2). To prove (1) and (2), the essential case is when E is a finite extension of the prime field; when E has positive characteristic (1) and (2) follow from Quillen's computation of the K-theory of finite fields [**Q2**]. For E a number field, (1) and (2) are the conjectures of Lichtenbaum and Quillen in the case of K_3, i.e. if E is a number field, the Chern class

$$c_{2,1}\colon K_3(E)^{\mathrm{ind}} \otimes \mathbf{Z}_l \to H^1_{\text{ét}}(E, \mathbf{Z}_l(2))$$

is an isomorphism. Merkurjev and Suslin have obtained these results, using similar methods. Here we give a sketch of the proof of Hilbert's Theorem 90 for relative K_2, and its application to the Lichtenbaum-Quillen conjecture for K_3.

Let R be a semilocal PID with Jacobson radical I. Let $\mathcal{D}$ be an Azumaya algebra over R, and X the associated Brauer-Severi scheme over R. Let $\overline{X}$ denote the fiber over $\overline{R} := R/I$. There is an E_1 spectral sequence converging to the relative K-theory $K_*(X, \overline{X})$ analogous to the Quillen spectral sequence converging to $K_*(X)$; the E_2 term $E_2^{p,q}(X, \overline{X})$ is a relative analogue

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