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CONFORMAL GEOMETRY AND COMPLETE MINIMAL SURFACES

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1. Introduction. This note applies new ideas from conformal geometry to the study of complete minimal surfaces of finite total curvature in R^3. The two main results illustrate this in complementary ways. Theorem A implies several uniqueness or nonexistence corollaries, while Theorem B constructs new examples, including the first complete immersed *nonorientable* minimal surfaces with finite total curvature and embedded ends. This construction is based upon the close relationship between these minimal surfaces and certain compact immersed surfaces minimizing the conformally invariant functional $W = \int H^2 da$ (Theorems C and D).

2. Main results. Let $M \,\ell\, R^3$ be an immersed complete minimal surface with finite total curvature $c(M) = \int_M K\, da$. Then (reinterpreting classical results of [**13**]) there is associated to M a $C^{1,\alpha}$ compact (possibly branched at ∞) immersed surface $\overline{M} \,\ell\, S^3 \sim R^3 \cup \infty$ such that $\overline{M} - \infty \sim M$. (The conformal equivalence $S^3 \sim R^3 \cup \infty$ is via stereographic projection.)

Denote by μ_x the number of sheets of $\overline{M}$ through a point $x \in S^3$, and set $\mu(M) = \max\{\mu_x | x \in R^3\}$, and $\eta(M) = \mu(\infty)$. Observe that $\eta(M)$ equals the number of ends of M iff each end is embedded iff $\overline{M} \,\ell\, S^3$ is unbranched [**9**].

THEOREM A. *$\mu(M) \leq \eta(M)$; equality implies M is a union of planes.*

COROLLARY 1. (i) *If $\eta(M) = 1$, then M is a plane* [**7**].

(i) *If $\eta(M) = 2$ and M is connected, then M is embedded.* (*This permits a simpler proof that M is a catenoid* [**14**].)

Assume now that M is connected and that each end is embedded. Express each end as *graph*(h) where $h \simeq cr^2 \log r$ is the height over its tangent plane at ∞ (in terms of inverted polar coordinates). An end is *flat* if $c = 0$; otherwise, it is a *catenoid* end. By viewing ∞ as a *triple-point* of $\overline{M} \,\ell\, S^3$ (called *transverse* if each small perturbation of $\overline{M}$ also has a triple-point) and using the relation [**1**] between the Euler number $\chi(\overline{M})$ and the number of triple-points, deduce

COROLLARY 2. *If $\eta(M) = 3$, then either*
(i) *$\chi(\overline{M})$ is even and the ends of M are not transverse; or*
(ii) *$\chi(\overline{M})$ is odd and the ends of M are transverse.*

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