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ALGEBRAIC VECTOR BUNDLES OVER REAL ALGEBRAIC VARIETIES

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By an affine algebraic variety, we mean in this note a locally ringed space $(X, \mathcal{R}_X)$ which is isomorphic to a ringed space of the form $(V, \mathcal{R}_V)$, where V is a Zariski closed subset in $\mathbf{R}^n$ and $\mathcal{R}_V$ is the sheaf of rings of regular functions on V. Recall that $\mathcal{R}_V(V)$ is the localization of the ring of polynomial functions on V with respect to the multiplicatively closed subset consisting of functions vanishing nowhere on V [**2**, **15**].

Let $\mathbf{F}$ be one of the fields $\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$ (quaternions). A continuous $\mathbf{F}$-vector bundle ξ over X is said to admit an algebraic structure if there exists a finitely generated projective module P over the ring $\mathcal{R}_X(X) \otimes_{\mathbf{R}} \mathbf{F}$ such that the $\mathbf{F}$-vector bundle over X, associated with P in the standard way, is C^0 isomorphic to ξ.

Our purpose is to study the following

PROBLEM. Characterize continuous $\mathbf{F}$-vector bundles over X which admit an algebraic structure.

This is an old problem, but despite considerable effort, the situation is well understood only in a few special cases: when X is the unit sphere S^n [**4**, **16**], when X is the real projective space $\mathbf{R}P^n$ [**5**, **7**] and when $\dim X \leq 3$ and $\mathbf{F} = \mathbf{R}$ [**8**, **9**] (cf. also [**13**] for a short survey).

Clearly, $\mathbf{R}P^n$ with its natural structure of an abstract real algebraic variety is actually an affine variety and every affine real algebraic variety admits a locally closed embedding in some $\mathbf{R}P^n$.

Let us first consider $\mathbf{C}$-vector bundles.

Let X be an affine nonsingular real algebraic variety and assume for a moment that X is embedded in $\mathbf{R}P^n$ as a locally closed subvariety. Consider $\mathbf{R}P^n$ as a subset of the complex projective space $\mathbf{C}P^n$. Let U be a Zariski neighborhood of X in the set of nonsingular points of the Zariski (complex) closure of X in $\mathbf{C}P^n$. Denote by $H^{\text{even}}_{\text{alg}}(U, \mathbf{Z})$ the subgroup of the cohomology group $H^{\text{even}}(U, \mathbf{Z})$ generated by the cohomology classes which are Poincaré dual to the homology classes in the Borel-Moore homology group $H_{\text{even}}(U, \mathbf{Z})$ represented by the closed irreducible complex algebraic subvarieties of U (cf. [**3**]). Let $H^{\text{even}}_{\mathbf{C}-\text{alg}}(X, \mathbf{Z})$ be the image of $H^{\text{even}}_{\text{alg}}(U, \mathbf{Z})$ via the restriction homomorphism $H^{\text{even}}(U, \mathbf{Z}) \to H^{\text{even}}(X, \mathbf{Z})$. One easily checks that $H^{\text{even}}_{\mathbf{C}-\text{alg}}(X, \mathbf{Z})$ does not depend on the choice of U or the choice of the embedding of X in $\mathbf{R}P^n$.

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