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RADEMACHER-TYPE FORMULAS FOR THE MULTIPLICITIES OF IRREDUCIBLE HIGHEST-WEIGHT REPRESENTATIONS OF AFFINE LIE ALGEBRAS

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The weight lattice of an integrable irreducible highest-weight representation of an affine Lie algebra is a union of infinite strings. Furthermore, the multiplicities in a finite number of strings determine all the multiplicities. Kac and Peterson [**1, 2**] have shown that the multiplicities in the same string are the Fourier coefficients of a modular form of negative weight, called a string function. This result, together with combinatorial identities for the Dedekind η-function, was successfully used in [**1** and **2**] to determine the string functions in many interesting cases. In this paper, we make use of the result of [**1** and **2**] to adapt Rademacher's circle method to the string functions and derive formulas for their coefficients. The formulas obtained are of the type proved by Rademacher in [**3**] for the partition function. This new approach opens the way towards an explicit determination of the Fourier coefficients which goes beyond the combinatorial method. As an example, we show how to calculate the coefficients in the case of the affine Lie algebra of type $C_4^{(1)}$, which is the simplest nontrivial case not treated in [**1** and **2**].

1. String functions. Our main purpose here is to state a version of the transformation law for string functions obtained by Kac and Peterson [**1, 2**]. We refer the reader to the original papers for details.

Let $\overline{\mathfrak{g}}$ be a complex simple Lie algebra. Let $\overline{\mathfrak{h}}$ be a Cartan subalgebra of $\overline{\mathfrak{g}}$ and denote by $\overline{\Delta}$ the root system of $(\overline{\mathfrak{g}}, \overline{\mathfrak{h}})$. Then $\overline{\mathfrak{g}} = \overline{\mathfrak{h}} \oplus (\bigoplus_{\alpha \in \overline{\Delta}} \overline{\mathfrak{g}}_\alpha)$, where $\overline{\mathfrak{g}}_\alpha = \{X \in \overline{\mathfrak{g}} \mid [H, X] = \alpha(H)X \text{ for all } H \in \overline{\mathfrak{h}}\}$. We fix a set of positive roots $\overline{\Delta}_+$ with simple roots $\alpha_1, \ldots, \alpha_l$. Let θ be the highest root of $\overline{\Delta}_+$. Let $(X, Y) = 2gB(X, Y)$ for all $X, Y \in \overline{\mathfrak{g}}$, where B is the Killing form of $\overline{\mathfrak{g}}$ and $g = (B(\theta, \theta))^{-1}$.

We denote by $\mathfrak{g}$ the (untwisted) *affine Lie algebra* associated with $\overline{\mathfrak{g}}$. That is, $\mathfrak{g} = (\mathbf{C}[t, t^{-1}] \otimes \overline{\mathfrak{g}}) \oplus \mathbf{C}c \oplus \mathbf{C}d$, where c is central and we define

$$[P \otimes X, Q \otimes Y] = PQ \otimes [X, Y] + \operatorname{Res}\left(\frac{dP}{dt} Q\right)(X, Y)c,$$

$$[d, P \otimes X] = t\frac{dP}{dt} \otimes X, \quad \text{and} \quad [d, d] = 0,$$

for all $P, Q \in \mathbf{C}[t, t^{-1}]$, $X, Y \in \overline{\mathfrak{g}}$. $\mathfrak{h} = \overline{\mathfrak{h}} \oplus \mathbf{C}c \oplus \mathbf{C}d$ is the *Cartan subalgebra* of $\mathfrak{g}$. We extend $(\ ,\ )$ to $\mathfrak{h} \times \mathfrak{h}$ by setting $(H, c) = (H, d) = 0$ for all $H \in \mathfrak{h}$,

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