BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 16, Number 2, April 1987

EXPONENTIAL SUMS AND NEWTON POLYHEDRA

ALAN ADOLPHSON AND STEVEN SPERBER

Let p be a prime number and let k denote the field of $q = p^a$ elements. Fix a nontrivial additive character $\Psi\colon k \to \mathbf{Q}(\varsigma_p)^\times$. Given a variety V of dimension n and a regular function f on V, with both V and f defined over k, one can define an exponential sum

$$(1) \qquad S(V, f) = \sum_{x \in V(k)} \Psi(f(x)),$$

where $V(k)$ denotes the k-rational points of V. It is a classical problem to find conditions on V and f that will imply a good estimate for $|S(V, f)|$. By "good estimate" we mean an inequality of the form

$$(2) \qquad |S(V, f)| \leq C\sqrt{q}^{\,n},$$

where C is a constant depending on V and f but not on q.

Deligne's fundamental theorem [3] reduces the problem of estimating the archimedean size of exponential sums to the problem of computing certain associated l-adic cohomology groups. Let $\mathbf{A}^n$ denote affine n-space over k and let $(\mathbf{G}_m)^n$ denote the product of n copies of the multiplicative group $\mathbf{G}_m$ over k. The purpose of this note is to report on some general criteria, when $V = (\mathbf{G}_m)^n$ or $\mathbf{A}^n$, that allow us to calculate this cohomology and hence obtain sharp archimedean estimates for the corresponding exponential sums. These same criteria allow us to obtain apparently sharp p-adic estimates for the exponential sums as well, although space limitations prevent us from describing them here. Connections between the p-adic theory and Newton polyhedra already appear in [7 and 8].

A novel feature of our work is the use of Dwork cohomology [4, 5] to compute l-adic cohomology. The results of this note have not so far been obtainable by purely l-adic methods. Complete proofs and references will appear elsewhere. We are indebted to B. Dwork and N. Katz for many helpful discussions.

1. Statement of results. Let k_r denote the extension of k of degree r and let $\mathrm{Tr}_r\colon k_r \to k$ be the trace map. Let $\bar{k}$ denote the algebraic closure of k. Set

$$(3) \qquad S_r(V, f) = \sum_{x \in V(k_r)} \Psi(\mathrm{Tr}_r\, f(x)),$$

Received by the editors November 1, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11L40; Secondary 14F20, 14F30.

First author partially supported by NSF Grant No. DMS-8401723; second author partially supported by NSF Grant No. DMS-8301453.

0273-0979/87 \$1.00 + \$.25 per page