BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 16, Number 2, April 1987

MORSE THEORY FOR FIXED POINTS OF SYMPLECTIC DIFFEOMORPHISMS

ANDREAS FLOER

ABSTRACT. We prove the following special case of the Arnold conjecture on the fixed points of an exact deformation φ of a compact closed symplectic manifold P: If $\pi_2(P) = 0$ and all fixed points of φ are nondegenerate, then their number is greater than or equal to the sum of the Betti numbers of P with respect to Z_2 coefficients.

Let P be a symplectic manifold, i.e. P is a smooth manifold equipped with a closed and nondegenerate 2-form ω. Then we can assign to each smooth function

$$H: P \times \mathbf{R} \to \mathbf{R}; \quad H(x,t) = H_t(x) \tag{1}$$

a family X_t of vector fields on P defined by $\omega(\cdot, X_t) = dH_t$. This vector field is called the (exact) Hamiltonian vector field associated with the (time-dependent) Hamiltonian H. If P is compact, then the differential equation

$$\frac{d}{dt}\varphi_{H,t}(x) = X_t(\varphi_{H,t}(x)) \tag{2}$$

with initial condition $\varphi_{H,0}(x) = x$ defines a family of smooth diffeomorphisms of P, which also preserve the symplectic structure, i.e. for each $t \in \mathbf{R}$ we have $\varphi_t^*\omega = \omega$. In fact, the set

$$\mathcal{D} = \{\varphi_{H,t} |\ t \in \mathbf{R} \text{ and } H \in C^\infty(P \times \mathbf{R})\} \tag{3}$$

of exact diffeomorphisms turns out to be a subgroup of the group of symplectic diffeomorphisms on P.

Since each $\varphi \in \mathcal{D}$ is homotopic to the identity, the Lefschetz fixed point theorem implies that if all fixed points x of φ are nondegenerate in the sense that

$$\det(D\varphi(x) - \mathrm{id}) \neq 0, \tag{4}$$

then the sum of the signs of (4) over all fixed points of φ is equal to the Euler characteristic $\chi(P)$. In particular, if all fixed points are nondegenerate, their number must be equal to or greater than the absolute value of $\chi(P)$. It has been conjectured by V. Arnold that a stronger result holds for exact diffeomorphisms: the number of fixed points of each $\varphi \in \mathcal{D}$ should satisfy estimates similar to those obtained by Morse theory for the number of critical points of a smooth function on P.

Received by the editors August 1, 1986 and, in revised form, November 5, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C15; Secondary 58F05.