MONOPOLES ON ASYMPTOTICALLY EUCLIDEAN 3-MANIFOLDS

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ABSTRACT. We consider a generalization of Yang-Mills-Higgs theory on Euclidean $\mathbb{R}^3$ to connected sums of $\mathbb{R}^3$ with compact closed 3-manifolds.

In this note, we describe progress in Yang-Mills-Higgs theory on 3-dimensional Riemannian manifolds. In particular, we are interested in the set of minima of the Yang-Mills action

$$\mathfrak{A}(A, \Phi) = \int_M \left( |F_A(x)|^2 + |\nabla_A \Phi(x)|^2 \right) d\mu(x),$$

where the "Higgs field" $\Phi$ is a section of a metric vector bundle $E$ over $M$ and $A$ is a linear connection on $E$ preserving the metric. For simplicity, we will restrict ourselves to the case where $E = \text{ad}(P)$ is the adjoint bundle of an $\text{SU}_2$-principal bundle $P$ over $M$.

The functional $\mathfrak{A}$ has been studied in great detail in the case where $M$ is the Euclidean $\mathbb{R}^3$, see [8]. We recall here briefly the main results: The length of the Higgs field $\Phi$ of any finite action configuration $c = (A, \Phi)$ obtains in some sense an asymptotic value $m(c)$ at infinity. For each $m > 0$, the space of finite action configurations $c$ with $m(c) = m$ decomposes into a family of components indexed by a "topological charge" $k$. On each of these components, the minima of the action functional (1) can be shown to be solutions of the Bogomolny equation

$$b_{\pm}(c) = \nabla_A \Phi \mp *F_A = 0$$

with the sign equal to the sign of $k$. Since reversing the sign of $\Phi$ changes the sign of $k$ while leaving $\mathfrak{A}$ invariant, we can restrict ourselves to the case $k > 0$ and write $b = b_+$. The set of gauge equivalence classes of solutions of (2), also called monopoles, is a smooth manifold $M_k$ of dimension $4k$. It can be described by means of algebraic geometry, see [7 and 3]. One usually considers the $(4k - 1)$-dimensional submanifolds $M_k^m$ of monopoles $[c]$ with fixed "mass" $m(c) = m$. In fact, one can without loss of generality set $m = 1$, since a scaling involving a dilation of $\mathbb{R}^3$ shows that $M_k^m \cong M_k^1$ for all $m > 0$.

In order to generalize these ideas, we replace $\mathbb{R}^3$ by an asymptotically flat manifold $M$, see [9]. This means that $M$ is the connected sum of $\mathbb{R}^3$ with a compact manifold, equipped with a metric which at the end of $M$ is a perturbation of the Euclidean metric in a certain sense. Since in this