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# ON THE TIME-OPTIMALITY OF BANG-BANG TRAJECTORIES IN $\mathbf{R}^3$

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**1. Introduction.** We study the problem of time-optimal control for a system

$$\Sigma\colon \dot{x} = f(x) + u \cdot g(x), \qquad |u| \leq 1,\ x \in \mathbf{R}^3,\ u \in \mathbf{R}$$

where $f$ and $g$ are smooth vector fields. Admissible controls are arbitrary measurable functions with values in $[-1, +1]$ and a trajectory of the system corresponding to a control $u(\cdot)$ is an absolutely continuous curve such that $\dot{x}(t) = f(x(t)) + u(t)g(x(t))$ holds almost everywhere.

In every optimal control problem the question of regularity of the solutions comes up naturally. The standard existence theorems only prove the existence of an optimal solution within the class of measurable functions. Necessary conditions for optimality put certain restrictions on optimal controls, in particular on their regularity properties. But, in general, they do not exclude a pathological behavior of optimal controls. In principle, the sets of discontinuities of an optimal control could even be a Cantor-like set with positive measure. On the other hand, certain regularity properties are necessary to construct a regular synthesis, i.e. to obtain sufficient conditions. For this one of the essential properties needed is that for every compact set $K$ there exists an integer $N = N(K)$ such that any time-optimal trajectory that lies in $K$ is a concatenation of at most $N$ "nice" pieces [**3**].

For our problem "nice" simply means trajectories corresponding to the constant controls $u = +1$ and $u = -1$ (bang arcs) or to a singular control (singular arc), which is a control usually with values in the interior of the control set and which satisfies certain compatibility conditions. (These are the possible candidates for time-optimal controls to which the necessary conditions of the Pontryagin maximum principle lead.)

For smooth single-input control-linear systems in the plane Sussmann showed that generically every point has a neighborhood $U$ such that time-optimal trajectories that lie in $U$ are finite concatenations of bang and singular arcs with a bound on the number of pieces [**4**]. This was the key element in the proof of the existence of a regular synthesis for basically arbitrary analytic systems of this form in the plane [**5, 6**]. A major issue in the proof of these results is to exclude the optimality of bang-bang trajectories with a large number of switchings in small times. (A bang-bang trajectory is a concatenation of bang arcs.) Since Sussmann's argument depended heavily on being in $\mathbf{R}^2$, even for the three-dimensional case so far only partial results

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