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SINGULAR LOCI OF SCHUBERT VARIETIES FOR CLASSICAL GROUPS

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In this note, we give an explicit description of the singular locus of a Schubert variety in the flag variety G/B, where G is a classical group, and B a Borel subgroup of G.

Let G be a classical group, and T a maximal torus in G. Let W be the Weyl group, and R the system of roots, of G relative to T. Let B be a Borel subgroup of G, where $B \supset T$. Let S (resp. R^+) be the set of simple (resp. positive) roots of R relative to B. For $\alpha \in R$, let s_α be the reflection with respect to α, and X_α the element in the Chevalley basis for the Lie algebra of G, associated to α. For $w \in W$, let $e(w)$ denote the point in G/B corresponding to w. The Schubert variety $X(w)$, where $w \in W$, is by definition the Zariski closure of $B\ e(w)$ in G/B. ($X(w)$ is understood to be endowed with the canonical reduced structure.) Let $\succeq$ denote the Bruhat order in W. It is well known that for $w_1, w_2 \in W$,

$$w_1 \succeq w_2 \quad \text{if and only if} \quad X(w_1) \supseteq X(w_2).$$

(For generalities on algebraic groups, one may refer to [**1**].)

The results on the singular locus of a Schubert variety are obtained as consequences of "standard monomial theory" as developed in *Geometry of* G/P. I–V (cf. [**11, 7, 4, 5, 8**]). One of the consequences of standard monomial theory is the First Basis Theorem (cf. [**5, 8, 6**]) which gives a $\mathbf{Z}$ basis

$$\{P(\lambda,\mu),\ (\lambda,\mu) \text{ an admissible pair}\}$$

for $H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L_{\mathbf{Z}})$, where $P_{\mathbf{Z}}$ is a maximal parabolic subgroup scheme of $G_{\mathbf{Z}}$ and $L_{\mathbf{Z}}$ is the ample generator of $\operatorname{Pic}(G_{\mathbf{Z}}/P_{\mathbf{Z}})$. For any field k, let us denote the canonical image of $P(\lambda,\mu)$ in $H^0(G_{\mathbf{Z}} \otimes k/P_{\mathbf{Z}} \otimes k, L_{\mathbf{Z}} \otimes k)$ by $p(\lambda,\mu)$. In [**9**], it is shown that over any field k, for $w, \tau \in W$, with $w \succeq \tau$, the Zariski tangent space $T(w,\tau)$, to $X(w)$ at $e(\tau)$ is spanned by

$$\left\{ X_{-\beta}, \beta \in \tau(R^+) \,\middle|\, \begin{array}{l} \text{for all } (\lambda,\mu) \text{ such that } X_{-\beta}p(\lambda,\mu) = cp(\tau,\tau), c \in k^*, \\ p(\lambda,\mu)|_{X(w)} \not\equiv 0 \end{array} \right\}.$$

Denoting by $\{Q(\lambda,\mu)\}$ the basis for the $\mathbf{Z}$-dual of $H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L_{\mathbf{Z}})$, dual to the basis $\{P(\lambda,\mu)\}$, it can be seen easily that $X_{-\beta}p(\lambda,\mu) = cp(\tau,\tau)$, $c \in k^*$, if and only if $X_{-\beta}Q(\tau,\tau)$, when written as a $\mathbf{Z}$-linear combination of the elements

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