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ON THE REAL SPECTRUM OF A RING AND ITS APPLICATION TO SEMIALGEBRAIC GEOMETRY

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Introduction. This paper is meant as an introduction and a guide to some recent developments in real algebraic geometry — more precisely, in semialgebraic geometry. In real algebraic geometry one is concerned with the set of real points $V(\mathbf{R})$ of a variety V defined over $\mathbf{R}$. More generally, one may replace the field of real numbers $\mathbf{R}$ by any real closed field. Real algebraic geometry is clearly a part of general algebraic geometry and therefore there seems to be no need for special considerations, i.e. special notions, tools, etc. However, in dealing with the set of real points $V(\mathbf{R})$ one encounters new phenomena which are not, or at least not easily, treatable by the general methods of algebraic geometry. To give examples, let V be an affine variety over $\mathbf{R}$. Then $V(\mathbf{R})$ can be regarded as an algebraic subset of some suitable $\mathbf{R}^N$, i.e., a subset defined by a finite set of polynomial equations $F_1 = 0, \ldots, F_r = 0$ where $F_i \in \mathbf{R}[X_1, \ldots, X_N]$, $i = 1, \ldots, r$. Consequently, $V(\mathbf{R})$ carries the subspace topology inherited from $\mathbf{R}^N$. Even if V is irreducible it may happen that $V(\mathbf{R})$ is not a connected topological space. Note that the corresponding set of complex points $V(\mathbf{C})$ is always connected if V is irreducible. A typical example is provided by the elliptic curve E (Figure 1).

In this example, $E(\mathbf{R})$ has two components C_1, C_2, namely

$$C_1 = \{(x, y) \in \mathbf{R}^2 \mid y^2 = x(x^2 - 1),\ x \leqslant 0\},$$

$$C_2 = \{(x, y) \in \mathbf{R}^2 \mid y^2 = x(x^2 - 1),\ x \geqslant 1\},$$

We notice that the components are described by equalities and inequalities. This is quite generally true: $V(\mathbf{R})$ always has a finite number of components each of which can be described by a finite number of polynomial equalities and inequalities, cf. [**Lo, Wh**].

Thus, one is naturally led to consider subsets of $V(\mathbf{R})$ which can be described by finitely many polynomial equalities and inequalities: these are the so-called semialgebraic subsets of $V(\mathbf{R})$.

Semialgebraic subsets of $V(\mathbf{R})$ or $\mathbf{R}^N$ arise in the above-mentioned study of components. However, they are to be considered as the natural objects of study in real algebraic geometry not only because of this occurrence. Their definition takes account of the entire structure of the real numbers as an ordered field.

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