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THE RIEMANN HYPOTHESIS FOR HILBERT SPACES OF ENTIRE FUNCTIONS

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History of the scattering method. The scattering theory of linear differential operators is an approach to their spectral analysis which arises from the study of wave propagation. The medium in which waves are propagated is regarded as a linear system whose input comes from the remote past and whose output is given in the distant future. The scattering operator describes the energy-preserving transition from past to future.

The awareness of a relationship between scattering theory and that area of number theory associated with the Riemann hypothesis is at least twenty-five years old. It is found for example in an analysis of the Laplace-Beltrami operator by Ehrenpreis and Mautner [4] and in a congress address of Gelfand [6]. In the same years the author constructed the Hilbert spaces of entire functions which are used in the present formulation of scattering theory.

In 1972, Faddeev and Pavlov [5] applied the Lax-Phillips scattering theory to the Laplace-Beltrami operator, considered in a space of functions invariant under the action of the modular group. An account of their result in English, and a generalization, appeared in 1976 in a monograph by Lax and Phillips [7]. The important observation is made that the Riemann hypothesis is equivalent to decay properties in the wave propagation associated with the operator. But no geometric reason could be found for the propagation to have these

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