BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 14, Number 2, April 1986

GENERALIZATIONS OF THE NEUMANN SYSTEM

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0. Introduction. The following observation, due to E. Trubowitz [**7**], illustrates an intimate relationship between spectral theory and Hamiltonian mechanics in the presence of constraints. Let $q(s)$ be a real periodic function such that Hill's operator,

$$L = \left(\frac{d}{ds}\right)^2 - q(s),$$

has only a finite number g of simple eigenvalues. There exist $g+1$ periodic eigenfunctions $x_0, \ldots, x_g$ and corresponding eigenvalues $a_0, \ldots, a_g$ of L such that

$$1 = \sum_{r=0}^{g} x_r^2 \quad \text{and} \quad q = -\sum_{r=0}^{g} (a_r x_r^2 + y_r^2),$$

where $y_r = dx_r/ds$. The equations $Lx_r = a_r x_r$ $(r = 0, \ldots, g)$ are equivalent to the classical Neumann system [**7**].

H. Flaschka [**3**] obtained similar results from a different point of view. His approach is based on the articles [**2** and **5**] of I. V. Cherednik and I. M. Krichever. The familiar Lax pairs, the constants of motion and the quadrics of the Neumann system emerge as consequences of the Riemann-Roch Theorem.

The purpose of our work is to apply Flaschka's techniques to operators of order $n \geq 2$. We will be defining higher Neumann systems whose theory is closely tied to the spectral theory of linear differential operators of order n. C. Tomei [**9**], using scattering theory, obtained some of our $n = 3$ formulas.

Preliminaries.

(1.1) RIEMANN SURFACE. Let R be a Riemann surface of genus g_R with a point ∞ and a rational function whose divisor of poles $(\lambda)_\infty$ is n^∞. We set $\kappa = \lambda^{1/n}$. Then κ^{-1} is a local parameter vanishing at ∞. Let W be the set of Weierstrass gap numbers of ∞.

(1.2) ALGEBRAIC CURVES. We assume that R admits a second rational function z with the following 3 properties. There exists an integer $N \geq 0$ and an integer $l \in \{1, 2, \ldots, n-1\}$ relatively prime to n such that

$$z = \lambda^{-N} \kappa^{-l} (z_0 + z_1 \kappa^{-1} + \cdots), \qquad z_0 = 1, \text{ at } \infty.$$

Let $(z)_\infty = (0) + \cdots + (m)$, $(r) \in R$, be the divisor of poles of z. Let $a_r = \lambda(r)$. We assume that each (r) is a simple pole and $a_r \neq a_s$ whenever $s \neq r$. We

Received by the editors September 30, 1985.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58F07, 58F19, 14H40.

[1]Supported in part by NSF (Fellowship) Grant No. MCS-8211308, NSF Grant No. MCS-8102748, and Department of the Army DAAG 29-82-K-0068.

0273-0979/86 $1.00 + $.25 per page