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THERE ARE ASYMPTOTICALLY FAR FEWER POLYTOPES THAN WE THOUGHT

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The problem of enumerating convex polytopes with n vertices in $\mathbf{R}^d$ has been the object of considerable study going back to ancient times (see [**4**, §13.6] for some remarks about the history of this problem since the nineteenth century). While significant progress has been made when the number of vertices was not too much larger than the dimension [**4**], little had been known above dimension 3 in the general case despite considerable efforts: if $f(n,d)$ is the *logarithm* of the number of combinatorially distinct simplicial polytopes with n vertices in $\mathbf{R}^d$, the sharpest asymptotic bounds previously known for $f(n,d)$ were

$$c_1 n \log n < f(n,d) < c_2 n^{d/2} \log n,$$

with the lower bound due to Shemer [**7**], and the upper bound following from the (asymptotic) Upper Bound Theorem of Klee [**5**], leaving a wide gap between the two bounds. (Here, and in the sequel, we take the view that d is fixed and $n \to \infty$; thus all constants depend on d.) The purpose of the present note is to announce a considerable narrowing of this gap:

THEOREM. $c_1 n \log n < f(n,d) < c_3 n \log n$.

The proof of the new upper bound is based on results of Milnor [**6**] and Goodman-Pollack [**2**]. An outline follows; details, as well as a related result on the number of combinatorial equivalence classes of labelled point configurations, will appear in [**3**].

Step 0. Instead of considering isomorphism classes of simplicial polytopes, we consider isomorphism classes of *labelled* simplicial polytopes, i.e., polytopes whose vertices are numbered, modulo only those isomorphisms of the simplicial structure which respect the numbering. This introduces a factor of $n!$ (or less, depending on the order of the symmetry group), hence—by Stirling's formula—does not affect the result.

Step 1. We suppress the simplicial structure of each polytope under consideration, and consider only the *order type* of its set of vertices $P_1, \ldots, P_n$; the order type of a configuration of n points in $\mathbf{R}^d$ is its equivalence class under the relation

$$\{P_1, \ldots, P_n\} \sim \{Q_1, \ldots, Q_n\}$$

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