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GAUSS' CLASS NUMBER PROBLEM FOR IMAGINARY QUADRATIC FIELDS

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1. Early history. In 1772 Euler [**11**] thought it noteworthy to remark that

$$x^2 - x + 41 = \text{prime}, \qquad x = 1, 2, \ldots, 40.$$

This subject was again touched upon by Legendre [**28**] in 1798 when he announced

$$x^2 + x + 41 = \text{prime}, \qquad x = 0, 1, \ldots, 39.$$

These remarkable polynomials, which take on prime values for many values of x, are one of the earliest recorded instances of a phenomenon related to what is now commonly referred to as Gauss' class number one problem. In fact, at the Fifth International Congress of Mathematicians, Rabinovitch [**34**] stated the following

THEOREM (RABINOVITCH). $D < 0$, $D \equiv 1 \pmod 4$,

$$x^2 - x + \frac{1 + |D|}{4} = \textit{prime}, \qquad x = 1, 2, \ldots, \frac{|D| - 3}{4},$$

if and only if every integer of the field $Q(\sqrt{D})$ has unique factorization into primes.

A similar theorem holds for the polynomial $x^2 + x + (1 + |D|)/4$. It is known that $Q(\sqrt{-163})$ has the unique factorization property, and this accounts for the remarkable polynomials above.

Gauss' class number problem has a long, curious, and interesting history. Perhaps the subject really goes back to Fermat, who in 1654 stated theorems like (here p = prime)

$$p = 6n + 1 \Rightarrow p = x^2 + 3y^2,$$
$$p = 8n + 1 \Rightarrow p = x^2 + 2y^2,$$

which were first proved by Euler in 1761 and 1763. Many other representation theorems of integers as sums of squares were proved in the eighteenth century,

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