BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

0273-0979/85 $1.00 + $.25 per page

Mathematical scattering theory, by H. Baumgärtel and M. Wollenberg, Akademie-Verlag, Berlin, 1983, 449 pp., 98 DM.

Apparently it was Leonardo da Vinci who first mentioned the diffraction of light. By this he meant the illumination observed within the geometrical shadow of an opaque body. Huygens, then Young and Fresnel in the early nineteenth century, proposed that this was due to the interference effects of different parts of the light rays. Subsequently, other diffraction—or scattering—phenomena were discovered in acoustics, elasticity, quantum mechanics, and so on. A great impetus came from Maxwell's theory of electromagnetism. Mathematical formulations were given in classical times by Helmholtz, Kirchhoff, Rayleigh, and Sommerfeld.

What do these disparate physical situations have in common? First, all of them are described by a kind of wave equation. Second, you have an incident wave which comes in and gets disturbed or diffracted or scattered. What comes out is the reflected or refracted wave. The scattering process takes you from the incident wave u_{in} to the reflected wave u_{out}. It could happen that part of the incident wave is "trapped" or "bound" instead of scattered. The picture to keep in mind is that of a billiard ball bouncing off an irregularly shaped obstacle. It could bounce off cleanly or it could get trapped within an indentation of the obstacle. In order to describe the mathematical context, we have to become more specific.

EXAMPLE 1. In electromagnetism the basic equations are Maxwell's. Raleigh demonstrated that the reason the sky is blue is that light scatters off the water droplets in the atmosphere. But let's simplify and just take the ordinary wave equation. Say the incident wave is $u_{\text{in}}(x, t) = \exp[i\lambda(t - \omega \cdot x)]$ and the scatterer is an opaque body $B \subset \mathbf{R}^3$. Then we must solve

$$u_{tt} - \Delta u = 0 \quad \text{outside } B \tag{1}$$

with an appropriate boundary condition and an initial condition given by u_{in} for $t \ll 0$. We need to decompose the solution of (1) into harmonic plane waves (i.e., do a Fourier analysis), as well as take a limit as $t \to +\infty$. The scattering matrix tells you how much of the incident wave of frequency λ and direction ω goes into the reflected wave of frequency λ' and direction ω'. Part of the wave may be trapped. The wave-ray duality which goes back 150 years plays a prominent role. It is enjoying a revival with the advent of "microlocal analysis".

EXAMPLE 2. In two-body quantum theory one of the bodies is placed at the origin, so that only the state of the other body matters. This state is described by the Schrödinger equation

$$-i\frac{\partial u}{\partial t} = -\Delta u(x, t) + V(x)u(x, t), \qquad x \in \mathbf{R}^3, \tag{2}$$