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THE TODA FLOW ON A GENERIC ORBIT IS INTEGRABLE

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ABSTRACT. For the generic orbit of the coadjoint action of the lower triangular group on its dual Lie algebra, we exhibit a complete set of integrals in involution for the associated Toda flow.

The Toda flow [**1–3**] on an orbit of the coadjoint action of the identity component L of the lower triangular group on its dual Lie algebra $\{S\colon S = S^T\}$ is generated by the Hamiltonian $\frac{1}{2}\operatorname{tr} S^2$. As is well known [**1–5**], the eigenvalues of S provide integrals in involution for the flow. In particular, the flow on an orbit consisting of tridiagonal matrices is completely integrable. The purpose of this note is to describe sufficient additional integrals to prove that the Toda flow on a generic coadjoint orbit is also integrable.

For an $n \times n$ matrix M, define the polynomial

$$\begin{aligned} P_k(M,\lambda) &\equiv \det\{(M-\lambda)_{ij}\colon k+1 \le i \le n,\ 1 \le j \le n-k\}, \\ &= \sum_{r=0}^{n-2k} E_{r,k}(M)\lambda^{n-2k-r}, \qquad 0 \le k \le \left[\frac{n}{2}\right]. \end{aligned}$$

The signs of $E_{0,k}(S)$ are preserved under the coadjoint action of L. We say that an orbit O_S through S is *generic* if $E_{0,k}(S)$ is nonzero, $0 \le k \le [n/2]$. On a generic orbit define

$$I_{r,k}(S) = E_{r,k}(S)/E_{0,k}(S), \qquad 0 \le k \le [(n-1)/2],\ 1 \le r \le n-2k.$$

THEOREM 1. *The generic orbit O_S is the set $\{U = U^T\colon \operatorname{sgn} E_{0,k}(U) = \operatorname{sgn} E_{0,k}(S),\ 0 \le k \le [n/2],\ I_{1,k}(U) = I_{1,k}(S),\ 0 \le k \le [(n-1)/2]\}$ and has dimension $2[n^2/4]$. The functions $I_{r,k}(S)$, $0 \le k \le [(n-1)/2]$, $2 \le r \le n-2k$, provide $[n^2/4]$ integrals in involution for the Toda flow. Furthermore, the integrals are independent on a dense open set in O_S.*

Let $\mathrm{GL}_k(n)$, $0 \le k \le [(n-1)/2]$, denote the identity component of the group of invertible matrices obtained by setting equal to zero all entries which are strictly above the main diagonal and lie in the first k rows or the last k columns. A key fact in the proof of Theorem 1 is that each $I_{r,k}$ is invariant under the restriction to $\mathrm{GL}_k(n)$ of the coadjoint action of $\mathrm{GL}(n)$ on its dual Lie algebra, i.e., $I_{r,k}(g^T M (g^T)^{-1}) = I_{r,k}(M)$ for all $g \in \mathrm{GL}_k(n)$ and all matrices M. In particular, as $L \subset \mathrm{GL}_k(n)$, it follows as in the Kostant-Symes theorem (see e.g. [**3**]) that the Poisson brackets for the $I_{r,k}$'s on O_S can be computed in $T^*(\mathrm{GL}(n))$, where the invariance under the full group $\mathrm{GL}_k(n)$ is then used to prove involution. Also the flows on the orbit O_S are projections of the associated flows on $T^*(\mathrm{GL}(n))$. Indeed, we have

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