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HOPF BIFURCATION IN THE PRESENCE OF SYMMETRY

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In this note we state a generalization of the Hopf bifurcation theorem to differential equations with symmetry. We state the results for ordinary differential equations although they apply, via standard reduction techniques (see Marsden and McCracken [**1976**]), to certain partial differential equations as well. Consider the ordinary differential equation

$$(1) \qquad dz/dt + F(z,\lambda) = 0 \qquad (F\colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n \text{ is } C^\infty),$$

where $F(0,\lambda) = 0$ and F commutes with an (orthogonal) action of a compact Lie group Γ on $V = \mathbf{R}^n$; that is, $F(\gamma x, \lambda) = \gamma F(x,\lambda)$ for $\gamma \in \Gamma$. Assume $dF|_{0,0}$ has pure imaginary eigenvalues. The symmetry can force these eigenvalues to have high multiplicity and the standard Hopf theorem does not apply. Despite this degeneracy, the symmetry can also force the occurrence of a branch of periodic solutions to (1).

Interactions between Hopf-type bifurcation and symmetry have been studied previously by several authors. Ruelle [**1973**] deals mainly with bifurcations of mappings, Schecter [**1976**] analyzes the continuous case. Rand [**1982**] and Renardy [**1982**] mainly consider bifurcations to tori from rotating waves. Schecter [**1976**], Bajaj [**1982**], Van Gils [**1984**], and Chossat and Iooss [**1984**] all consider the example $\Gamma = \mathbf{O}(2)$. Our approach differs from these by emphasizing the general role of isotropy subgroups in determining the occurrence of branches.

These ideas prove useful in studying Taylor-Couette flow of a fluid between coaxial rotating cylinders. See Chossat and Iooss [**1984**] and Golubitsky and Stewart [**1984b**]. Other potential applications include systems of identical coupled chemical oscillators (see Alexander and Auchmuty [**1984**]).

For $x \in V$ define the *isotropy group* $\Sigma_x = \{\sigma \in \Gamma | \sigma x = x\}$. Let $\Sigma \subseteq \Gamma$ and define the *fixed-point subspace* $V^\Sigma = \{y \in V | \sigma y = y \text{ for all } \sigma \in \Sigma\}$. Notice that F maps V^Σ to itself.

In order for $dF|_{0,0}$ to have pure imaginary eigenvalues, the representation of Γ on V must satisfy certain conditions. There are two 'simplest' cases:

(a) The action of Γ on V is irreducible but not absolutely irreducible.

(b) $V = W \oplus W$, where Γ acts absolutely irreducibly on W and by the diagonal action on $W \oplus W$.

Henceforth we assume (b) holds. If $dF|_{0,0}$ has pure imaginary eigenvalues then we may assume, without loss of generality, that $dF|_{0,0} = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$. The eigenvalues of $dF|_{0,\lambda}$ are $\sigma(\lambda) \pm i\phi(\lambda)$, each of multiplicity $\dim W = n/2$.

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