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COMPOSITION AND GENERA OF NORM-TYPE FORMS

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M. Kneser has recently discovered a way to define a composition of binary quadratic forms in general [5]. His basic idea can be described as expanding the structure to include a specified action of a ring of similitudes. This approach avoids the traditional problem of "orienting" the forms, since the "proper equivalences" can be defined simply as the isometries that preserve the action of the similitude ring. But more is true: when we view his idea in this way, we can extend it to norm-type forms of higher degrees. Besides throwing a new light on the quadratic case, this extension reveals a natural concept of genus underlying the "genus fields" already known in number theory.

Fix a base ring R (commutative with unit). If P is a free R-module, then a "form" of degree m supported by P is of course a homogeneous polynomial f of degree m in the coordinates on P; technically, this means that f is an element of the symmetric power $S^m(P^*)$, and in this version we can (and do) extend the definition to projective P of finite rank. Carrying over the usual terminology for quadratic forms, we call (P, f) *primitive* if f is not identically zero modulo any maximal ideal of R.

Now fix an extension C of R, and assume that C is projective of rank m as an R-module. A *form of type* C/R will be a pair (P, f) where

(1) P is an invertible C-module (and hence projective of rank m over R),

(2) f is a primitive form of degree m on the R-module P, and

(3) there is a formal identity $f(cp) = N(c)f(p)$, where N is the norm from C to R.

Two such forms are equivalent if there is a form isometry preserving the C-module structure. Let $F(C/R)$ be the set of equivalence classes. If $R \to S$ is any ring homomorphism, then $\otimes_R S$ induces a map $F(C/R) \to F(C \otimes S/S)$. There is always at least one form of type C/R, the *trivial* form (C, N), and in a sense this is the basic one:

THEOREM 1. *For any form of type C/R there is a faithfully flat $R \to S$ such that the form becomes trivial after extension to S.*

This is proved in two steps, first making the C-module free and then making the value of f on a generator into an mth power. Several results then follow by descent theory (see for instance [7]):

COROLLARY 1. *The classes $F(C/R)$ correspond to the (flat) cohomology classes in $H^1(R, \mathrm{Aut}(C, N))$.*

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