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The geometry of Coxeter groups, by H. Hiller, Research Notes in Mathematics, Vol. 54, Pitman Advanced Publishing Program, Boston, 1982, 213 pp., $21.95. ISBN 0-2730-8517-4

Quotients of Lie groups appear throughout mathematics and physics. To an algebraist, the nicest Lie groups are the simple Lie groups over the complex numbers. The most interesting quotients formed from these groups are the *flag manifolds* G/P obtained by dividing out by a "parabolic" subgroup. The algebraic geometric study of flag manifolds reveals several interesting interactions between algebraic geometry, representation theory, and combinatorics. The main topic of the book under review is the description of the cohomology rings of complex flag manifolds with an accent on the combinatorial aspects thereof.

By far the most famous of the flag manifolds are the Grassmannians. A Grassmannian $G_d(V)$ consists of the set of all d-dimensional subspaces of an n-dimensional complex vector space V together with a suitable topology. To express $G_d(V)$ in the form G/P, first note that the group of linear transformations with determinant 1, the special linear group $\mathrm{SL}(V)$, acts transitively on d-dimensional subspaces. Pick any d-dimensional subspace W, and let P be the subgroup of $G = \mathrm{SL}(V)$ which stabilizes W. Then G/P with the quotient topology is the desired Grassmannian. More generally, let $W_1 \subset W_2 \subset \cdots \subset W_k$ be a strictly increasing sequence of subspaces of V (a *flag*), and let P be the subgroup of G which stabilizes these subspaces. Most often a maximal flag ($k = n$) is taken, and then P is a "Borel" subgroup B. The resulting manifold G/B is sometimes referred to as *the* flag manifold.

Grassmannians can be generalized in a second direction. Let G be the special orthogonal group $\mathrm{SO}(V)$ or symplectic group $\mathrm{Sp}(V)$. These are the subgroups of $\mathrm{SL}(V)$ which preserve symmetric or antisymmetric bilinear forms, respectively. Then the parabolic subgroups are again stabilizers of (suitably defined) flags of subspaces. All the simple Lie groups are known. Up to simple connectedness, there are only five other complex simple Lie groups: E_6, E_7, E_8, F_4, and G_2. Flag manifolds can be formed from these groups using the general definition of parabolic subgroup given below.

What is a Coxeter group? There is a general definition, but Hiller's book is mainly concerned with finite Coxeter groups. Ignoring the dihedral groups, there are only two finite Coxeter groups which are not *Weyl* groups. Let E be an n-dimensional Euclidean space. A Weyl group is a finite subgroup of the orthogonal group $\mathrm{O}(E)$ which is generated by n reflections and which leaves an n-dimensional lattice of points in E invariant (hence the chemists' terminology: "point crystallographic groups"). Irreducible Weyl groups have been classified. There are 3 infinite families: the symmetries of the regular n-simplex (the symmetric group S_n), the symmetries of the n-cube, and a certain index 2 subgroup of the cube group. And again there are five exceptional cases.