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THE GÖDEL CLASS WITH IDENTITY IS UNSOLVABLE

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The Gödel Class with Identity (GCI) is the class of closed, prenex formulas of pure quantification theory extended by inclusion of the identity sign "=" whose prefixes have the form $\forall x \forall y \exists z_0 \cdots \exists z_n$. At the end of [2], Gödel claims that the GCI can be shown to contain no infinity axioms—and hence to be decidable (for satisfiability)—"by the same method" as he employed to show this for the analogous class without identity. (An infinity axiom is a satisfiable formula that has no finite models.) Gödel's claim has been questioned for almost twenty years; since no obvious extension of Gödel's method seemed to apply to the GCI, the decision problem for this class has been deemed open. Gödel's claim is, in fact, erroneous; below we explicitly construct an infinity axiom F in the GCI. Moreover, by exploiting further properties of F, we can encode an undecidable problem into the GCI. Hence the GCI is undecidable.

The formula F contains the monadic predicate letter Z and the dyadic letters S, P_1, P_2, Q, N, R_1, R_2. F is designed so that, in every model $\mathcal{M}$ of F, there will be a unique element $\overline{0}$ such that $\mathcal{M} \models Z\overline{0}$, a unique element $\overline{1}$ such that $\mathcal{M} \models S\overline{1}\,\overline{0}$, a unique element $\overline{2}$ such that $\mathcal{M} \models S\overline{2}\,\overline{1}$, and so on *ad infinitum*. Thus Z acts as the predicate "is zero", and S as the successor relation. The other letters are used to insure the existence of such $\overline{0}, \overline{1}, \overline{2}, \ldots$, and are meant to act as follows. Elements of $\mathcal{M}$ can be taken to encode pairs of integers. Suppose b encodes $\langle p, q\rangle$; then P_1 holds between b and the element $\overline{p}$, P_2 between b and $\overline{q}$, Q between b and $\overline{q+1}$, N between b and an element that encodes $\langle p+1, q\rangle$, R_1 between b and any element that encodes $\langle q+1, r\rangle$ for some r, and R_2 between b and any element that encodes $\langle r, q+1\rangle$ for some r.

Let F be a prenex form of $\forall x \forall y \exists z_0 H$, where H is the conjunction of the following ten clauses:

$$(1) \qquad Zx \wedge Zy \to x = y;$$

$$(2) \qquad Zz_0 \wedge \neg Sz_0x \wedge \bigwedge_{\delta=1,2} (P_\delta x z_0 \wedge P_\delta xy \to y = z_0);$$

$$(3) \qquad (\exists z)Szx;$$

$$(4) \qquad \neg Zx \wedge x \neq y \to (\exists w)(Sxw \wedge -Syw);$$

$$(5) \qquad Sxy \to (\exists z)(Qzx \wedge P_2zy \wedge P_1zz_0);$$

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