REAL AND COMPLEX CHEBYSHEV APPROXIMATION
ON THE UNIT DISK AND INTERVAL

BY MARTIN H. GUTKNECHT AND LLOYD N. TREFETHEN

We announce the resolution of a number of outstanding questions regarding real and complex Chebyshev (supremum norm) approximation by rational functions on a disk and on an interval. The proofs consist mainly of symmetry arguments applied to explicit examples. The most important results: complex rational best approximations on a disk are in general not unique; real functions on an interval can in general be approximated arbitrarily much better by complex rational functions than by real ones. Details will appear in [3, 8].

1. Notation. Define $\Delta = \{z : |z| \leq 1\}$, $A_\Delta = \{f : \text{continuous on } \Delta, \text{analytic in the interior}\}$, $||f||_\Delta = \sup \{|f(z)| : z \in \Delta\}$. Let $m \geq 0$, $n \geq 1$ be integers (all questions considered below become trivial for $n = 0$), and let $R_{mn}$ be the space of complex rational functions of type $(m, n)$. Define $A'_{\Delta} = \{f \in A_\Delta : f(z) = \overline{f(\overline{z})}\}$, $R'_{mn} = \{r \in R_{mn} : r(z) = \overline{r(\overline{z})}\}$, and for $f \in A_{\Delta}$,

$$E_{mn}(f; \Delta) = \inf_{r \in R_{mn}} ||f - r||_\Delta, \quad E'_{mn}(f; \Delta) = \inf_{r \in R'_{mn}} ||f - r||_\Delta.$$

It is known that these infima are attained (proof by a normal families argument due to Walsh [10]), and we let $N_{mn}(f; \Delta)$ and $N'_{mn}(f; \Delta)$ denote the number (finite or infinite) of best approximations (BA’s) to $f$.

Finally, set $I = [-1, 1]$, and let $A_I$, $A'_I$, $|| \cdot ||_I$, $E_{mn}(f; I)$, $E'_{mn}(f; I)$, $N_{mn}(f; I)$, $N'_{mn}(f; I)$ be defined analogously. ($A_I$ and $A'_I$ are just the sets of continuous complex and real functions on $I$, respectively.)

2. Nonuniqueness. It is a classical result due to Achieser that $N'_{mn}(f; I) = 1$ for all $m$, $n$ and all $f \in A'_I$. But Lungu [4] (on proposal of A. A. Gončar) and independently Saff and Varga [6, 7] found that for all $m$ and $n$ there exists $f \in A'_I$ with $E_{mn}(f; I) < E'_{mn}(f; I)$, so that by symmetry necessarily $N_{mn}(f; I) \geq 2$. Ruttan [5] even gave an example with $N_{11}(f; I) = \infty$. However, the analogous questions for the disk have been open [2, 9]. We claim [3]:

**Theorem 1.** $\forall m, n, \forall K \geq 1, \exists f \in A_\Delta$ such that $N_{mn}(f; \Delta) \geq K$.

**Theorem 2.** $\forall m, n$ with $m = 0$ or $n = 1$, $\exists f \in A'_\Delta$ such that $E_{mn}(f; \Delta) < E'_{mn}(f; \Delta)$.

**Theorem 3.** $\forall m, n, \exists f \in A'_\Delta$ such that $N'_{mn}(f; \Delta) > 1$. 

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