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REAL AND COMPLEX CHEBYSHEV APPROXIMATION ON THE UNIT DISK AND INTERVAL

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We announce the resolution of a number of outstanding questions regarding real and complex Chebyshev (supremum norm) approximation by rational functions on a disk and on an interval. The proofs consist mainly of symmetry arguments applied to explicit examples. The most important results: complex rational best approximations on a disk are in general not unique; real functions on an interval can in general be approximated arbitrarily much better by complex rational functions than by real ones. Details will appear in [**3**, **8**].

1. Notation. Define $\Delta = \{z\colon |z| \le 1\}$, $A_\Delta = \{f\colon$ continuous on Δ, analytic in the interior$\}$, $\|f\|_\Delta = \sup\{|f(z)|\colon z \in \Delta\}$. Let $m \ge 0$, $n \ge 1$ be integers (all questions considered below become trivial for $n = 0$), and let R_{mn} be the space of complex rational functions of type (m,n). Define $A^r_\Delta = \{f \in A_\Delta\colon f(\bar z) = \overline{f(z)}\}$, $R^r_{mn} = \{r \in R_{mn}\colon r(\bar z) = \overline{r(z)}\}$, and for $f \in A_\Delta$,

$$E_{mn}(f;\Delta) = \inf_{r \in R_{mn}} \|f - r\|_\Delta, \qquad E^r_{mn}(f;\Delta) = \inf_{r \in R^r_{mn}} \|f - r\|_\Delta.$$

It is known that these infima are attained (proof by a normal families argument due to Walsh [**10**]), and we let $N_{mn}(f;\Delta)$ and $N^r_{mn}(f;\Delta)$ denote the number (finite or infinite) of *best approximations* (*BA's*) to f.

Finally, set $I = [-1,1]$, and let A_I, A^r_I, $\|\cdot\|_I$, $E_{mn}(f;I)$, $E^r_{mn}(f;I)$, $N_{mn}(f;I)$, $N^r_{mn}(f;I)$ be defined analogously. (A_I and A^r_I are just the sets of continuous complex and real functions on I, respectively.)

2. Nonuniqueness. It is a classical result due to Achieser that $N^r_{mn}(f;I) = 1$ for all m, n and all $f \in A^r_I$. But Lungu [**4**] (on proposal of A. A. Gončar) and independently Saff and Varga [**6**, **7**] found that for all m and n there exists $f \in A^r_I$ with $E_{mn}(f;I) < E^r_{mn}(f;I)$, so that by symmetry necessarily $N_{mn}(f;I) \ge 2$. Ruttan [**5**] even gave an example with $N_{11}(f;I) = \infty$. However, the analogous questions for the disk have been open [**2**, **9**]. We claim [**3**]:

THEOREM 1. *$\forall m, n, \forall K \ge 1, \exists f \in A_\Delta$ such that $N_{mn}(f;\Delta) \ge K$.*

THEOREM 2. *$\forall m, n$ with $m = 0$ or $n = 1$, $\exists f \in A^r_\Delta$ such that $E_{mn}(f;\Delta) < E^r_{mn}(f;\Delta)$.*

THEOREM 3. *$\forall m, n, \exists f \in A^r_\Delta$ such that $N^r_{mn}(f;\Delta) > 1$.*

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