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ON THE ZEROS OF DIRICHLET SERIES ASSOCIATED WITH CERTAIN CUSP FORMS[1]

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As is well known, in 1859 Riemann [6] conjectured that the function $\varsigma(s)$ defined in $\operatorname{Re} s > 1$ by the Dirichlet series $\sum_{n=1}^{\infty} n^{-s}$ has all its zeros, apart from the "trivial" zeros at the negative even integers, on the line $\operatorname{Re} s = \frac{1}{2}$. It is known that these "nontrivial" zeros lie symmetrically about the line $\operatorname{Re} s = \frac{1}{2}$ within the strip $0 < \operatorname{Re} s < 1$. The truth of this Riemann Hypothesis would have a profound impact in the theory of numbers, particularly with regard to the distribution of primes.

One of the major achievements in this theory was due to Selberg [7] in 1943. He proved for $\varsigma(s)$ that a positive proportion of the nontrivial zeros lie on the critical line. Later authors have given specific numerical values for this proportion. In this note we announce the proof of a similar theorem for Dirichlet series attached to certain cusp forms on the full modular group. We formulate the specific theorem below.

Let $F(z)$ be a holomorphic cusp form of even integral weight k and constant multiplier system for the full modular group $\Gamma(1) = SL(2, \mathbf{Z})/\{\pm I\}$. That is,

$$F(Mz) = (cz+d)^k F(z), \qquad M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma(1),$$

and $F(z)$ vanishes at $i\infty$. Expand $F(z)$ in a "Fourier series" at the cusp $i\infty$ as

$$F(z) = \sum_{l=1}^{\infty} f(l) e^{2\pi i l z}.$$

The Dirichlet series $L_f(s) = \sum_{l=1}^{\infty} f(l) l^{-s}$ converges absolutely for

$$\operatorname{Re} s > (k+1)/2$$

and can be continued to an entire function in the s-plane. Furthermore, $L_f(s)$ has all its nontrivial zeros in the strip $(k-1)/2 < \operatorname{Re} s < (k+1)/2$. Let

$$N(T) = \#\{\rho = \beta + i\gamma \colon 0 < \gamma < T, (k-1)/2 < \beta < (k+1)/2, L_f(\rho) = 0\}$$

and

$$N_0(T) = \#\{\rho = k/2 + i\gamma \colon 0 < \gamma < T, L_f(\rho) = 0\}.$$

It is known [4] that $N(T) \sim cT \log T$ for some constant $c > 0$. We then have the following theorem.

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