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EXISTENCE THEOREMS FOR GENERALIZED KLEIN-GORDON EQUATIONS

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The semilinear elliptic partial differential equation

$$(1) \qquad Lu = f(x,u), \qquad x \in \Omega,$$

is to be considered in smooth unbounded domains $\Omega \subseteq R^N$, $N \geq 2$, where

$$(2) \qquad Lu = -\sum_{i,j=1}^{N} D_i[a_{ij}(x)D_j u] + b(x)u, \qquad x \in \Omega,$$

$D_i = \partial/\partial x_i$, $i = 1, \ldots, N$; each $a_{ij} \in C^{1+\alpha}_{\text{loc}}(\Omega)$, $b \in C^{\alpha}_{\text{loc}}(\Omega)$, $0 < \alpha < 1$; $b(x) \geq b_0 > 0$ for all $x \in \overline{\Omega}$, L is uniformly elliptic in Ω, and $f(x,u)$ satisfies all the conditions in either list (F) or list (F′) below. Our main objective is to prove the existence of a positive solution $u(x)$ of (1) in Ω satisfying the boundary condition $u(x) = 0$ on $\partial\Omega$ (void if $\Omega = R^N$), and to obtain asymptotic estimates as $|x| \to \infty$.

The physical importance of the Klein-Gordon prototype

$$(3) \qquad -\Delta u + b(x)u = \delta[p(x)u^{\gamma} - q(x)u^{\beta}], \qquad x \in \Omega,$$

arises in particular from nonlinear field theory; the existence of solitary waves and asymptotic behavior as $|x| \to \infty$ follow from our theorems. It is assumed in (3) that p and q are nonnegative, bounded, and locally Hölder continuous in Ω, $1 < \gamma < \beta$, and $\delta = \pm 1$. The Hypotheses (F′) below are all satisfied if $\delta = +1$ and p/q is bounded and bounded away from zero in Ω. Hypotheses (F) are all satisfied if $\delta = -1$, $\beta < (N+2)/(N-2)$, $N \geq 3$, and $q(x) > 0$.

HYPOTHESES F (UNBOUNDED NONLINEARITY)

(f_1) $f \in C^{\alpha}_{\text{loc}}(\Omega \times R)$ and $f(x,t)$ is locally Lipschitz continuous with respect to t for all $x \in \Omega$.

(f_2) There exist positive constants $s_i > 1$ and nonnegative, bounded continuous functions $f_i \in L^2\Omega$, $i = 1, \ldots, I$, such that

$$|f(x,t)| \leq \sum_{i=1}^{I} f_i(x)|t|^{s_i}, \qquad x \in \overline{\Omega},\ t \in R,$$

where each $s_i < (N+2)/(N-2)$ if $N \geq 3$.

(f_3) $f(x,t)/t \to +\infty$ as $t \to +\infty$ locally uniformly in Ω.

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